

MOMENTS OF L -FUNCTIONS AND THE LIOUVILLE-GREEN METHOD

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To the memory of Professor N.V. Kuznetsov

ABSTRACT. We show that the harmonic percentage of primitive forms of level one and weight $4k \rightarrow \infty$, $k \in \mathbf{N}$ for which the associated L -function at the central point is no less than $(\log k)^{-2}$ is at least 20 for an individual weight and at least 50 on average. The key ingredients of our proof are the Kuznetsov convolution formula and the Liouville-Green method.

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2010 *Mathematics Subject Classification.* Primary: 11F12; Secondary: 34E20.

Key words and phrases. central values of L -functions; non-vanishing; the Liouville-Green method; weight aspect; WKB approximation.

1. INTRODUCTION

Non-vanishing results for central values of L -functions in families have numerous applications, discovered, for example, in [4, 10, 12, 13, 25]. In particular, this paper is inspired by the work of Iwaniec and Sarnak [10], where they approached the problem of non-existence of Landau-Siegel zeros by studying the non-vanishing of automorphic L -functions at the critical point.

In the weight aspect they proved the following result. Let $H_{2k}(1)$ be the space of primitive forms of level 1 and weight $2k \geq 12$, $k \in \mathbf{N}$. For $f \in H_{2k}(1)$, let $L_f(1/2)$ be the associated L -function at the critical point. For any $\epsilon > 0$ one has

$$(1.1) \quad \lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k \leq K} \frac{\#\{f \in H_{4k}(1), \quad L_f(1/2) \geq 1/(\log k)^2\}}{\#\{f \in H_{4k}(1)\}} \geq \frac{1}{2} - \epsilon.$$

Moreover, the non-existence of Landau-Siegel zeros for Dirichlet L -functions of real primitive characters would follow if (1.1) is established with proportion strictly larger than $1/2$.

Inequality (1.1) was proved by Iwaniec and Sarnak [10] in 2000. Since then there were several attempts to obtain analogous result for an individual weight. Independently Fomenko [5] and Lau & Tsang [15] showed that the proportion of non-vanishing without extra average over weight is at least $1/\log k$. This is proved by computing the asymptotics of pure, unmollified first and second moments, and, therefore depends on k . Recently, Luo [16] showed that there is a strictly positive proportion of non-vanishing. However, his approach does not allow finding the exact proportion. The reason is that only an upper bound for the mollified second moment was proved in [16], while one requires to have full asymptotics in order to optimize the length of mollifier.

The main result of the present paper is the effective and strictly positive upper bound on the proportion of non-vanishing central L -values.

Theorem 1.1. *For any $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ such that for any $k \geq k_0$ and $k \equiv 0 \pmod{2}$ we have*

$$(1.2) \quad \sum_{\substack{f \in H_{2k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^h 1 \geq \frac{1}{5} - \epsilon.$$

The subscript h in the sum above means that the average is computed with the harmonic weight

$$\frac{\Gamma(2k-1)}{((4\pi)^{2k-1}\langle f, f \rangle_1)},$$

where $\langle f, f \rangle_1$ is the Petersson inner product on the space of level 1 holomorphic modular forms.

Our arguments rely in most part on the methods developed by Kuznetsov in 90's. More precisely, we prove the exact formulas for moments of L -functions in the critical strip and express the error for the first moment in terms of the confluent hypergeometric function ${}_1F_1(a, b; x)$ and the error for the second moment in terms of the Gauss hypergeometric function ${}_2F_1(a, b, c; x)$. The exact statements are given by Theorems 3.2 and 4.2, respectively.

The most challenging problem is to optimize estimates of the error terms for the second moment that are given by two shifted convolution sums, namely

$$\frac{1}{2\sqrt{l}} \sum_{n=1}^{l-1} \tau(n)\tau(l-n)\phi_k\left(\frac{n}{l}\right) + \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n)\tau(n+l)\Phi_k\left(\frac{l}{n+l}\right),$$

where $\tau(n)$ is the number of divisors function and

$$(1.3) \quad \phi_k(x) = 2 \left(-\log x - 2\frac{\Gamma'}{\Gamma}(k) + 2\frac{\Gamma'}{\Gamma}(1) \right) {}_2F_1(k, 1-k, 1; x) - \\ 2 \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2\frac{\partial}{\partial c} \right) {}_2F_1(a, b, c; x) \Big|_{\substack{a=k \\ b=1-k \\ c=1}},$$

$$(1.4) \quad \Phi_k(x) = 2 \frac{\Gamma^2(k)}{\Gamma(2k)} x^k {}_2F_1(k, k, 2k; x).$$

As the main tool we choose the Liouville-Green method (also called WKB approximation or the Liouville-Steklov method). It is one of the oldest approximation techniques widely applied, for example, in quantum mechanics. The idea of using it in analytic number theory belongs to Kuznetsov.

The method is based on the observation that "close" differential equations have "close" solutions. Accordingly, in section 5 we find differential equations satisfied by functions (1.3) and (1.4). The given equations can be approximated by other differential equations which have "simpler" functions as solutions. This allows approximating the

error terms uniformly in k with any power of precision. See Theorems 5.13 and 5.16.

The Liouville-Green approximation models the behavior of functions (1.3) and (1.4) using Y_0 and K_0 -Bessel functions with the large parameter k in the argument. This shows that in the required ranges $\Phi_k(x)$ decays exponentially and $\phi_k(x)$ is oscillatory. To smooth out the oscillations of $\phi_k(x)$ one can average the error terms over weight with a suitable test function, reproving the result of Iwaniec and Sarnak (1.1), as shown in Theorem 8.8.

Finally, asymptotic formulas for twisted moments have several other applications. For example, Hough [7] considered zero-density estimates for L -functions in the weight aspect. His proof is based on the asymptotic evaluation of the second moment near the critical line with the error term estimated as $O(l^{3/4}k^{-1/2+\epsilon})$ at the central point. The same error bound was obtained by Ng Ming Ho [20] using different approach. Our method (see Theorem 6.4) yields $O(l^{1/2}k^{-1/2+\epsilon})$.

2. NOTATIONS AND TECHNICAL LEMMAS

For $v \in \mathbf{C}$ let

$$(2.1) \quad \tau_v(n) = \sum_{n_1 n_2 = n} \left(\frac{n_1}{n_2} \right)^v = n^{-v} \sigma_{2v}(n),$$

where

$$(2.2) \quad \sigma_v(n) = \sum_{d|n} d^v.$$

Note that $\tau_v(n) = \tau_{-v}(n)$. Let $e(x) = \exp(2\pi i x)$. The classical Kloosterman sum

$$S(n, m; c) = \sum_{\substack{a \pmod{c} \\ (a, c) = 1}} e\left(\frac{an + a^*m}{c}\right), \quad aa^* \equiv 1 \pmod{c},$$

satisfies Weil's bound (see [27])

$$(2.3) \quad |S(m, n; c)| \leq \tau_0(c) \sqrt{(m, n, c)} \sqrt{c}.$$

For $\Re s > 1$, $m \geq 1$ (see [24, Eq. 1.5.4])

$$(2.4) \quad \sum_{c=1}^{\infty} \frac{S(0, m; c)}{c^s} = \frac{\sigma_{1-s}(m)}{\zeta(s)},$$

where $\zeta(s)$ is the Riemann zeta function.

Let $H_{2k}(1)$ be the set of primitive forms of level 1 and weight $2k \geq 12$. Every $f \in H_{2k}(1)$ has a Fourier expansion of the form

$$(2.5) \quad f(z) = \sum_{n \geq 1} \lambda_f(n) n^{(2k-1)/2} e(nz).$$

The Fourier coefficients of primitive forms are multiplicative

$$(2.6) \quad \lambda_f(m) \lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right).$$

For each $f \in H_{2k}(1)$ the associated L -function is defined by

$$(2.7) \quad L_f(s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s}, \quad \Re s > 1.$$

As a consequence of relation (2.6), for $\Re u > 1/2$, $\Re v = 0$ we have

$$(2.8) \quad \sum_{n=1}^{\infty} \frac{\tau_v(n) \lambda_f(n)}{n^{1/2+u}} = \frac{1}{\zeta(1+2u)} L_f(1/2+u+v) L_f(1/2+u-v).$$

Let $\Gamma(s)$ be the Gamma function. The completed L -function

$$(2.9) \quad \Lambda_f(s) = \left(\frac{1}{2\pi}\right)^s \Gamma\left(s + \frac{2k-1}{2}\right) L_f(s)$$

satisfies the functional equation

$$(2.10) \quad \Lambda_f(s) = \epsilon_f \Lambda_f(1-s), \quad \epsilon_f = i^{2k}$$

and can be analytically continued on the whole complex plane. It follows from equation (2.10) that $L_f(1/2) = 0$ for odd k .

The harmonic summation is defined by

$$(2.11) \quad \sum_{f \in H_{2k}(1)}^h \alpha_f := \sum_{f \in H_{2k}(1)} \frac{\Gamma(2k-1)}{(4\pi)^{2k-1} \langle f, f \rangle_1} \alpha_f,$$

where $\langle f, f \rangle_1$ is the Petersson inner product on the space of level 1 holomorphic modular forms.

We denote by $J_v(x)$, $Y_v(x)$, $K_v(x)$ the Bessel functions.

Theorem 2.1. (*Petersson's trace formula*, [22]) *For $2k \geq 12$ and integral $l, n \geq 1$ one has*

$$(2.12) \quad \sum_{f \in H_{2k}(1)}^h \lambda_f(l) \lambda_f(n) = \delta_{l,n} + 2\pi i^{2k} \sum_{c=1}^{\infty} \frac{S(l, n; c)}{c} J_{2k-1}\left(\frac{4\pi\sqrt{ln}}{c}\right).$$

Consider two Bessel kernels

$$(2.13) \quad k_0(x, v) = \frac{1}{2 \cos \pi(1/2 + v)} (J_{2v}(x) - J_{-2v}(x)),$$

$$(2.14) \quad k_1(x, v) = \frac{2}{\pi} \sin \pi(1/2 + v) K_{2v}(x).$$

Lemma 2.2. ([6, Eq. 6.561.14, 6.561.16, 17.43.16, 17.43.18]) *Let*

$$(2.15) \quad \gamma(u, v) = \frac{2^{2u-1}}{\pi} \Gamma(u+v) \Gamma(u-v).$$

For $3/2 > \Re w > 2|\Re v|$ one has

$$(2.16) \quad \int_0^\infty k_0(x, v) x^{w-1} dx = \gamma(w/2, v) \cos(\pi w/2),$$

and for $\Re w > 2|\Re v|$

$$(2.17) \quad \int_0^\infty k_1(x, v) x^{w-1} dx = \gamma(w/2, v) \sin(\pi(1/2 + v)).$$

Consider the series

$$(2.18) \quad D_v(s, x) := \sum_{n \geq 1} \frac{\tau_v(n)}{n^s} e(nx), \quad \Re v = 0, \quad \Re s > 1.$$

Let x be a rational number $x = \frac{d}{c}$ with $(d, c) = 1$, $c \geq 1$. Then the function $D_v(s, x)$ of two complex parameters s and v is meromorphic on the whole complex plane. If we fix v such that $\Re v = 0$ and $v \neq 0$, then $D_v(s, d/c)$ as a function of single variable s has two simple poles at $s = 1+v$ and $s = 1-v$ with residues $c^{-1-2v} \zeta(1+2v)$ and $c^{-1+2v} \zeta(1-2v)$, respectively, and it is regular elsewhere. Also it satisfies the functional equation (see [17, Lemma 3.7])

$$(2.19) \quad D_v\left(s, \frac{d}{c}\right) = \left(\frac{4\pi}{c}\right)^{2s-1} \gamma(1-s, v) \times \\ \left\{ -\cos \pi s D_v\left(1-s, -\frac{d^*}{c}\right) + \sin \pi(1/2 + v) D_v\left(1-s, \frac{d^*}{c}\right) \right\},$$

where $dd^* \equiv 1 \pmod{c}$ and $\gamma(u, v)$ is defined by (2.15). For $\Re s < 0$ the following estimate is satisfied (see [17, Eq. 3.3.24])

$$(2.20) \quad |D_v(s, d/c)| \ll (c|s|)^{1-2\Re s} (\log |s|)^2.$$

Note that

$$(2.21) \quad D_v(s, d/c) = \sum_{n=1}^{\infty} \frac{\tau_v(n)}{n^s} e\left(n \frac{d}{c}\right) = \frac{1}{c^{2s}} \sum_{a,b=1}^c e\left(ab \frac{d}{c}\right) \zeta(a/c, 0; s-v) \zeta(b/c, 0; s+v),$$

where

$$(2.22) \quad \zeta(\alpha, \beta; s) = \sum_{n+\alpha>0} \frac{e(n\beta)}{(n+\alpha)^s}, \quad \Re s > 1$$

is the Lerch zeta function. Applying the Euler-Maclaurin formula, one has (see [26, Lemma 3, p. 24])

$$(2.23) \quad \zeta(\alpha, 0; s) = \sum_{n=0}^N \frac{1}{(n+\alpha)^s} + \frac{1}{s-1} (N+1/2+\alpha)^{1-s} + s \int_{N+1/2}^{\infty} \frac{1/2 - \{u\}}{(u+\alpha)^{s+1}} du,$$

where $\{u\}$ is a fractional part of u . For any $\epsilon > 0$ we estimate the absolute value of (2.23), obtaining

$$(2.24) \quad |\zeta(\alpha, 0; \sigma + iT)| \ll_{\epsilon} \begin{cases} 1 & \sigma \geq 1 + \epsilon \\ \log T & 1 \leq \sigma \leq 1 + \epsilon \\ T^{1-\sigma} & \epsilon \leq \sigma < 1 \end{cases}.$$

The Mellin transform of function f is defined by

$$(2.25) \quad \hat{f}(z) = \int_0^{\infty} t^{z-1} f(t) dt.$$

Lemma 2.3. (*Parseval's inequality*) Assume that for some $a \in \mathbf{R}$

$$(2.26) \quad \int_0^{\infty} |g(x)| x^{-a} dx < \infty, \quad \int_{(a)} |\hat{\phi}(z)| dz < \infty.$$

Then

$$(2.27) \quad \frac{1}{2\pi i} \int_{(a)} \hat{\phi}(z) \hat{g}(1-z) dz = \int_0^{\infty} \phi(x) g(x) dx.$$

Proof. See, for example, [21, Section 3.1.3]. □

3. THE FIRST MOMENT

In this section we derive the asymptotic formula for the first moment of automorphic L -functions at the critical point.

Theorem 3.1. *For $2k \geq 12$, $l < k/4\pi e$*

$$(3.1) \quad \sum_{f \in H_{2k}(1)}^h \lambda_f(l) L_f(1/2) = l^{-1/2} (1 + i^{2k}) + O\left(\frac{1}{\sqrt{l}} \left(2\pi e \frac{l}{k}\right)^k\right).$$

This Theorem is a consequence of more general statement. Let us define two auxiliary functions

$$(3.2) \quad I_{\pm}(u, v, k, x) := e(\pm 1/8 \mp k/4) x^{1/2-k} \frac{\Gamma(k-v-u)}{\Gamma(2k)} \\ \times {}_1F_1\left(k-v-u, 2k; -\frac{e(\mp 1/4)}{x}\right)$$

and

$$(3.3) \quad V_1(l; u, v, k) := \sum_{c=1}^{\infty} \frac{1}{c^{1/2+u+v}} \sum_{\substack{n=1 \\ (n,c)=1}}^{\infty} \frac{e(n^* l c^{-1})}{n^{1/2-u-v}} (2\pi)^{u+v-1/2} \\ \times e\left(\frac{1/2-u-v}{4}\right) I_{-}\left(u, v, k, \frac{cn}{2\pi l}\right) + \sum_{c=1}^{\infty} \frac{1}{c^{1/2+u+v}} \sum_{\substack{n=1 \\ (n,c)=1}}^{\infty} \frac{e(-n^* l c^{-1})}{n^{1/2-u-v}} \\ \times (2\pi)^{u+v-1/2} e\left(-\frac{1/2-u-v}{4}\right) I_{+}\left(u, v, k, \frac{cn}{2\pi l}\right),$$

where $nn^* \equiv 1 \pmod{c}$.

Theorem 3.2. *For $2k \geq 12$, $\Re v = 0$, $|\Re u| < k-1$ we have*

$$(3.4) \quad \sum_{f \in H_{2k}(1)}^h \lambda_f(l) L_f(1/2 + u + v) = \frac{1}{l^{1/2+u+v}} \\ + i^{2k} \frac{(2\pi)^{2u+2v} \Gamma(k-u-v)}{l^{1/2-u-v} \Gamma(k+u+v)} + 2\pi i^{2k} V_1(l; u, v, k).$$

Furthermore, for $t \in \mathbf{R}$, $T = 1 + |t|$

$$(3.5) \quad V_1(l; 0, it, k) \ll \begin{cases} \frac{1}{\sqrt{lT}} \left(2\pi e \frac{lT}{k}\right)^k, & l < \frac{1}{4\pi e} \frac{k}{T} \\ l^{1/2} (lT)^{\epsilon} \max\left(\frac{\sqrt{T}}{k}, \frac{1}{\sqrt{k}}\right), & l \geq \frac{1}{4\pi e} \frac{k}{T}. \end{cases}$$

Proof. Our arguments follow the proof in [1]. The only difference is the second main term

$$i^{2k} \frac{(2\pi)^{2u+2v} \Gamma(k-u-v)}{l^{1/2-u-v} \Gamma(k+u+v)},$$

which appears due to an additional pole when level is equal to 1. \square

Taking $u = v = 0$ we obtain Theorem 3.1.

4. THE SECOND MOMENT

4.1. Voronoi's summation formula. Standard Voronoi's summation formulas (see [11]) are stated for a function ϕ with compact support. However, weaker conditions on ϕ are required to prove a convolution formula for the second moment.

Lemma 4.1. (*Kuznetsov, 1981*) *Let $\phi : [0, \infty) \rightarrow \mathbf{C}$ and its Mellin transform $\hat{\phi}(s)$ satisfy the following conditions:*

- (1) $\hat{\phi}(2s)$ is regular in the region $\sigma_0 < \Re s < \sigma_1$ for some $\sigma_1 > 1$ and $\sigma_0 < 0$;
- (2) for some $\epsilon > 0$ and for $\sigma_0 < \sigma < \sigma_1$ the function

$$\left((1 + |t|)^{1-2\sigma+\epsilon} + 1 \right) |\hat{\phi}(2\sigma + 2it)|$$

can be integrated on $(-\infty, +\infty)$.

Then for any v with $\Re v = 0$ and for any coprime integers $c, d \geq 1$ we have

$$(4.1) \quad \frac{4\pi}{c} \sum_{n=1}^{\infty} \tau_v(n) e\left(\frac{nd}{c}\right) \phi\left(\frac{4\pi\sqrt{n}}{c}\right) = 2 \frac{\zeta(1+2v)}{(4\pi)^{1+2v}} \hat{\phi}(2+2v) +$$

$$2 \frac{\zeta(1-2v)}{(4\pi)^{1-2v}} \hat{\phi}(2-2v) + \sum_{n=1}^{\infty} \tau_v(n) \times$$

$$\int_0^{\infty} \left(e\left(-\frac{nd^*}{c}\right) k_0(x\sqrt{n}, v) + e\left(\frac{nd^*}{c}\right) k_1(x\sqrt{n}, v) \right) \phi(x) x dx,$$

where $dd^* \equiv 1 \pmod{c}$.

Originally, Lemma 4.1 was proved by Kuznetsov in his doctoral thesis (1981) and also published in [14]. Unfortunately, book [14] is hard to find, so we provide all details here.

The proof of Lemma 4.1 is based on the properties of the series $D_v(s, x)$ defined by equation (2.18). By the inverse Mellin transform

$$\phi\left(\frac{4\pi\sqrt{n}}{c}\right) = \frac{1}{i\pi} \int_{(b)} \hat{\phi}(2s) \left(\frac{c}{4\pi}\right)^{2s} \frac{1}{n^s} ds, \quad 1 < b < \sigma_1.$$

Therefore,

$$\frac{4\pi}{c} \sum_{m=1}^{\infty} e\left(\frac{md}{c}\right) \tau_v(m) \phi\left(\frac{4\pi\sqrt{m}}{c}\right) = \frac{1}{i\pi} \int_{(b)} \left(\frac{c}{4\pi}\right)^{2s-1} \times \\ D_v\left(s, \frac{d}{c}\right) \hat{\phi}(2s) ds.$$

The change of order of integration and summation in the formula above is allowed since $\int_{(b)} |\hat{\phi}(2s)| ds < \infty$ and the series $D_v(s, d/c)$ converges absolutely for $\Re s = b > 1$.

Moving the contour of integration to $\Re s = \delta$ with $\sigma_0 < \delta < 0$, we cross two simple poles of $D_v(s, \frac{d}{c})$ at the points $1+v$ and $1-v$. Computation of the residues gives the first two summands on the right-hand side of (4.1). To justify this contour shift, we show that for

$$f(s) := \left(\frac{c}{4\pi}\right)^{2s-1} D_v\left(s, \frac{d}{c}\right) \hat{\phi}(2s)$$

one has

$$(4.2) \quad \int_{\Re s=b, |\Im s|>T} f(s) ds \rightarrow 0 \quad \text{as } T \rightarrow \infty,$$

$$(4.3) \quad \int_{\Re s=\delta, |\Im s|>T} f(s) ds \rightarrow 0 \quad \text{as } T \rightarrow \infty,$$

$$(4.4) \quad \int_{\delta}^b f(\sigma + iT) d\sigma \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Note that (4.2) is satisfied since

$$\int_T^{\infty} |\hat{\phi}(2b + 2iy)| dy \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

Property (4.3) follows from inequality (2.20) and

$$\int_{(\delta)} (|s| + 1)^{1-2\delta+\epsilon} |\phi(\hat{2s})| ds < \infty.$$

We split the integral in (4.4) into two parts $\int_{\delta}^{\epsilon} + \int_{\epsilon}^b$. For the first part we apply functional equation (2.19) and estimate everything by the absolute value using (2.21), (2.24). The second part is evaluated using expression (2.21) and estimates (2.24). This implies (4.4).

Finally, we compute

$$\frac{1}{i\pi} \int_{(\delta)} \left(\frac{c}{4\pi}\right)^{2s-1} D_v\left(s, \frac{d}{c}\right) \hat{\phi}(2s) ds$$

by applying functional equation (2.19). Since $\Re(1-s) > 1$, we switch the order of summation and integration, obtaining

$$\sum_{m \geq 1} \tau_v(m) \frac{1}{i\pi} \int_{(\delta)} \gamma(1-s, v) \hat{\phi}(2s) m^{s-1} \times \\ \left(-e\left(-\frac{ma}{c}\right) \cos \pi s + e\left(\frac{ma}{c}\right) \sin \pi(1/2 + v) \right) ds.$$

Now we can move the contour of integration to $\Re s = \alpha$ such that $3/4 < \alpha < 1$. Then the result follows from Lemmas 2.3 and 2.2, as we now show. Let

$$g_1(x) := x k_0(x\sqrt{m}, v),$$

then

$$\hat{g}_1(1-2s) = -\gamma(1-s, v) \cos(\pi s) m^{s-1}$$

and

$$-\frac{1}{i\pi} \int_{(\alpha)} \gamma(1-s, v) \cos \pi s \hat{\phi}(2s) m^{s-1} ds = \int_0^\infty k_0(x\sqrt{m}, v) \phi(x) x dx.$$

Parameter α is chosen such that condition (2.26) is satisfied for $g_1(x)$, i.e.

$$\begin{cases} 1 - 2\alpha > -1 & \text{as } x \rightarrow 0, \\ 1/2 - 2\alpha < -1 & \text{as } x \rightarrow \infty. \end{cases}$$

Similarly,

$$\frac{1}{i\pi} \int_{(\alpha)} \gamma(1-s, v) \sin \pi(1/2 + v) \hat{\phi}(2s) m^{s-1} ds = \\ \int_0^\infty k_1(x\sqrt{m}, v) \phi(x) x dx.$$

This concludes the proof of Lemma 4.1.

4.2. Convolution formula for the second moment.

Theorem 4.2. (*Kuznetsov, preprint 1994*) For $\Re v = 0$, $\Im v \neq 0$, $|\Re u| < k - 1$ we have

$$\begin{aligned}
(4.5) \quad M_2(l; u, v) &:= \sum_{f \in H_{2k}(1)}^h \lambda_f(l) L_f(1/2 + u + v) L_f(1/2 + u - v) = \\
&\tau_v(l) \left(\frac{\zeta(1 + 2u)}{l^{1/2+u}} + \frac{(2\pi)^{4u}}{l^{1/2-u}} \zeta(1 - 2u) \frac{\Gamma(k - u + v) \Gamma(k - u - v)}{\Gamma(k + u + v) \Gamma(k + u - v)} \right) + \\
&(-1)^k \tau_u(l) \frac{\zeta(1 + 2v)}{(2\pi)^{-2u+2v} l^{1/2+v}} \frac{\Gamma(k - u + v)}{\Gamma(k + u - v)} + \\
&(-1)^k \tau_u(l) \frac{\zeta(1 - 2v)}{(2\pi)^{-2u-2v} l^{1/2-v}} \frac{\Gamma(k - u - v)}{\Gamma(k + v + u)} + E(l; u, v).
\end{aligned}$$

The summand $E(l; u, v)$ can be expressed in terms of hypergeometric functions

$$\begin{aligned}
(4.6) \quad E(l; u, v) &= \frac{(-1)^k}{\sqrt{l}} \sum_{1 \leq n \leq l-1} \tau_v(n) \tau_u(l - n) \phi_k \left(\frac{n}{l}; u, v \right) + \\
&\frac{1}{\sqrt{l}} \sum_{n \geq l+1} \tau_v(n) \tau_u(n - l) \Phi_k \left(\frac{l}{n}; u, v \right) + \\
&\frac{(-1)^k}{\sqrt{l}} \sum_{n \geq 1} \tau_v(n) \tau_u(n + l) \psi_k \left(\frac{l}{n}; u, v \right),
\end{aligned}$$

where

$$(4.7) \quad \phi_k(x; u, v) = \tilde{\phi}_k(x; u, v) + \tilde{\phi}_k(x; u, -v),$$

$$\begin{aligned}
(4.8) \quad \tilde{\phi}_k(x; u, v) &= \frac{(2\pi)^{2u+1}}{2 \cos \pi(1/2 + v)} \frac{\Gamma(k - u + v)}{\Gamma(2v + 1) \Gamma(k + u - v)} \times \\
&x^v (1 - x)^{-u} {}_2F_1(k - u + v, 1 - k - u + v, 1 + 2v; x),
\end{aligned}$$

$$\begin{aligned}
(4.9) \quad \Phi_k(x; u, v) &= 2(2\pi)^{2u} \frac{\Gamma(k - u + v) \Gamma(k - u - v)}{\Gamma(2k)} \times \\
&\sin \pi(1/2 + u) x^k (1 - x)^{-u} {}_2F_1(k - u + v, k - u - v, 2k; x),
\end{aligned}$$

$$(4.10) \quad \psi_k(x; u, v) = 2(2\pi)^{2u} \frac{\Gamma(k-u+v)\Gamma(k-u-v)}{\Gamma(2k)} \times \\ \sin \pi(1/2+v)x^k(1+x)^{-u} {}_2F_1(k-u+v, k-u-v, 2k; -x).$$

Proof. We multiply both sides of the Petersson trace formula (2.12) by

$$n^{-1/2-u} \tau_v(n) \zeta(1+2u)$$

and sum over $n \geq 1$. Using relation (2.8), we obtain the first summand on the right-hand side of (4.5) plus the non-diagonal contribution

$$M_2(l; u, v) = \zeta(1+2u) \tau_v(l) l^{-1/2-u} + M^{ND},$$

$$M^{ND} = 2\pi i^{2k} \zeta(1+2u) \sum_{n \geq 1} \sum_{c \geq 1} \frac{S(l, n; c)}{c} \frac{\tau_v(n)}{n^{1/2+u}} J_{2k-1} \left(\frac{4\pi \sqrt{ln}}{c} \right).$$

Applying Weil's bound for Kloosterman sums and standard estimates for J -Bessel function, the double series can be estimated as follows

$$l^\epsilon \sum_{n \geq 1} \sum_{c \geq 1} c^{-1/2+\epsilon} n^{-\Re u - 1/2+\epsilon} \min \left(\left(\frac{\sqrt{ln}}{c} \right)^{2k-1}, \frac{\sqrt{c}}{(ln)^{1/4}} \right) \ll \\ l^{1/4+\epsilon} \sum_{n \geq 1} n^{-\Re u - 1/4+\epsilon}.$$

Thus for $\Re u > 3/4$ the series is absolutely convergent and we can change the order of summation in M^{ND} . Replacing Kloosterman sum by its definition in the inner series and applying Lemma 4.1 with

$$\phi(x) = x^{-1-2u} J_{2k-1}(x\sqrt{l}),$$

we obtain

$$M^{ND} = 2\pi i^{2k} (4\pi)^{2u} \zeta(1+2u) \sum_{c=1}^{\infty} \frac{1}{c^{1+2u}} \times \\ \left(2S(0, l; c) \frac{\zeta(1+2v)}{(4\pi)^{1+2v}} \hat{\phi}(2+2v) + 2S(0, l; c) \frac{\zeta(1-2v)}{(4\pi)^{1-2v}} \hat{\phi}(2-2v) + \right. \\ \sum_{n=1}^{\infty} \tau_v(n) S(0, l-n; c) \int_0^{\infty} k_0(x\sqrt{n}, v) \phi(x) x dx + \\ \left. \sum_{n=1}^{\infty} \tau_v(n) S(0, l+n; c) \int_0^{\infty} k_1(x\sqrt{n}, v) \phi(x) x dx \right).$$

Ramanujan's identity (2.4) and [6, Eq. 6.561(14)] for $-1/4 < \Re u < k$ allow expressing the first two terms in M^{ND} as the third and fourth

summands on the right-hand side of (4.5). The second summand in (4.5) comes from the third term in M^{ND} when $n = l$ by applying [6, Eq. 6.574(2)] for $0 < \Re u < k$.

Consider

$$2\pi i^{2k} (4\pi)^{2u} \zeta(1+2u) \sum_{c=1}^{\infty} \frac{1}{c^{1+2u}} \sum_{\substack{n=1 \\ n \neq l}}^{\infty} \tau_v(n) S(0, l-n; c) \times \\ \int_0^{\infty} k_0(x\sqrt{n}, v) \phi(x) x dx.$$

If $n < l$ we apply identity [6, Eq. 6.574] for $-1/2 < \Re u < k$ to evaluate the integral over x and Ramanujan's identity (2.4) to compute the sum over c , obtaining the first term in (4.6). Analogously, for $n > l$ we obtain the second term in (4.6) using [6, Eq. 6.574(1)]. Finally, the third term in (4.6) comes from

$$2\pi i^{2k} (4\pi)^{2u} \zeta(1+2u) \sum_{c=1}^{\infty} \frac{1}{c^{1+2u}} \sum_{n=1}^{\infty} \tau_v(n) S(0, l+n; c) \times \\ \int_0^{\infty} k_1(x\sqrt{n}, v) \phi(x) x dx$$

by applying [6, Eq. 6.576(3)] for $\Re u < k$ and identity (2.4).

Note that the left-hand side of convolution formula (4.5) is entire function of u and v . Since

$$|\tau_u(n \pm l)| \ll n^{|\Re u|+\epsilon}, \quad |\Phi_k(x; u, v)|, |\psi_k(x; u, v)| \ll x^k \text{ as } x \rightarrow 0,$$

the right-hand side of (4.5) is regular function for $|\Re v| + |\Re u| < k - 1$. \square

5. THE LIOUVILLE-GREEN METHOD

Our main references are chapter 12 of book [18] and paper [3]. We assume that k is an even positive integer.

5.1. Some properties of ϕ_k . In this section we study the function

$$\phi_k(x) = \lim_{\substack{u \rightarrow 0 \\ v \rightarrow 0}} \phi_k(x; u, v),$$

where $\phi_k(x; u, v)$ is defined by equation (4.7). Letting $u = 0$ and computing the limit as $v \rightarrow 0$ by L'Hospital's rule, we obtain

$$(5.1) \quad \phi_k(x) = \left. \frac{\partial}{\partial v} \left[\frac{-2\Gamma(k+v)x^v}{\Gamma(1+2v)\Gamma(k-v)} {}_2F_1(k+v, 1-k+v, 1+2v; x) \right] \right|_{v=0}.$$

Differentiation with respect to v gives

$$(5.2) \quad \phi_k(x) = 2 \left(-\log x - 2 \frac{\Gamma'}{\Gamma}(k) + 2 \frac{\Gamma'}{\Gamma}(1) \right) {}_2F_1(k, 1-k, 1; x) - \left. 2 \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2 \frac{\partial}{\partial c} \right) {}_2F_1(a, b, c; x) \right|_{\substack{a=k \\ b=1-k \\ c=1}}.$$

Lemma 5.1. *One has*

$$(5.3) \quad \phi_k(x) = (-1)^k \phi_k(1-x).$$

Proof. Recall that

$$\phi_k(x; u, v) = \tilde{\phi}_k(x; u, v) + \tilde{\phi}_k(x; u, -v),$$

where $\tilde{\phi}_k(x; u, v)$ is defined by equation (4.8). Applying [2, Eq. 33, p. 107] and Euler's reflection formula, we obtain

$$\begin{aligned} \tilde{\phi}_k(x; u, v) = & (-1)^k \frac{(2\pi)^{2u} \pi}{\sin \pi v} \left(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)} \times \right. \\ & \frac{\sin \pi(v+u)}{\Gamma(1-2u) \sin 2\pi u} x^v (1-x)^{-u} {}_2F_1(k+v-u, 1-k+v-u, 1-2u; 1-x) + \\ & \left. \frac{\sin \pi(v-u)}{\Gamma(1+2u) \sin(-2\pi u)} x^v (1-x)^u {}_2F_1(k+v+u, 1-k+v+u, 1+2u; 1-x) \right). \end{aligned}$$

Let

$$F(v, u) := x^v (1-x)^u {}_2F_1(k+v+u, 1-k+v+u, 1+2u; 1-x).$$

Then

$$\begin{aligned} \tilde{\phi}_k(x; u, v) = & (-1)^k \frac{(2\pi)^{2u} \pi}{2 \sin \pi u} \left(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)} \frac{F(v, -u)}{\Gamma(1-u)} - \right. \\ & \left. \frac{F(v, u)}{\Gamma(1+2u)} \right) + (-1)^k \frac{(2\pi)^{2u} \pi \cos \pi v}{2 \sin \pi v \cos \pi u} \times \\ & \left(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)} \frac{F(v, -u)}{\Gamma(1-2u)} + \frac{F(v, u)}{\Gamma(1+2u)} \right). \end{aligned}$$

Computing the limit as $u \rightarrow 0$ by L'Hospital rule, we have

$$\lim_{u \rightarrow 0} \tilde{\phi}_k(x; u, v) = (-1)^k \frac{\pi \cos \pi v}{\sin \pi v} F(v, 0) + \frac{(-1)^k}{2} \times$$

$$\left. \frac{\partial}{\partial u} \left(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)} \frac{F(v, -u)}{\Gamma(1-2u)} - \frac{F(v, u)}{\Gamma(1+2u)} \right) \right|_{u=0}.$$

By [2, Eq. 1-2, p. 105] it follows that

$$F(-v, 0) = F(v, 0).$$

Thus

$$\lim_{u \rightarrow 0} \left(\tilde{\phi}_k(x; u, v) + \tilde{\phi}_k(x; u, -v) \right) = \frac{(-1)^k}{2} \times$$

$$\left[\left. \frac{\partial}{\partial u} \left(\frac{\Gamma(k+v-u)\Gamma(k-v-u)}{\Gamma(k+v+u)\Gamma(k-v+u)} \frac{F(v, -u)}{\Gamma(1-2u)} - \frac{F(v, u)}{\Gamma(1+2u)} \right) \right|_{u=0} + \right.$$

$$\left. \frac{\partial}{\partial u} \left(\frac{\Gamma(k-v-u)\Gamma(k+v-u)}{\Gamma(k-v+u)\Gamma(k+v+u)} \frac{F(-v, -u)}{\Gamma(1-2u)} - \frac{F(-v, u)}{\Gamma(1+2u)} \right) \right|_{u=0} \right].$$

Letting $v = 0$, we have

$$\phi_k(x) = (-1)^k 4 \left(\frac{\Gamma'}{\Gamma}(1) - \frac{\Gamma'}{\Gamma}(k) \right) F(0, 0) - (-1)^k 2 F'_u(0, 0),$$

where

$$F'_u(0, 0) = \log(1-x) {}_2F_1(k, 1-k, 1; 1-x) +$$

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2 \frac{\partial}{\partial c} \right) {}_2F_1(a, b, c; 1-x) \Big|_{\substack{a=k \\ b=1-k \\ c=1}}.$$

Then formula (5.2) implies that $\phi_k(x) = (-1)^k \phi_k(1-x)$. \square

Corollary 5.2. *For any positive even integer k one has $\phi'_k(1/2) = 0$.*

Lemma 5.3. *The following series representation holds*

$$(5.4) \quad \phi_k(x) = {}_2F_1(k, 1-k, 1; x) 2 \log x - 2 \sum_{n=0}^{k-1} \frac{(-1)^n \Gamma(k+n)}{\Gamma(k-n) \Gamma^2(n+1)} \times \\ \left(-2 \frac{\Gamma'}{\Gamma}(n+1) + \frac{\Gamma'}{\Gamma}(k+n) + \frac{\Gamma'}{\Gamma}(k-n) \right) + \\ 2(-1)^k \sum_{n=k}^{\infty} \frac{\Gamma(n+k) \Gamma(n-k+1) x^n}{\Gamma^2(n+1)}.$$

Proof. By equation (5.1) and Euler's reflection formula

$$\phi_k(x) = -\frac{2}{\pi} \times \\ \frac{\partial}{\partial v} \left(x^v \sum_{n=0}^{\infty} \frac{\sin \pi(k-v) \Gamma(1-k+v+n) \Gamma(k+v+n)}{\Gamma(1+2v+n)} \frac{x^n}{n!} \right) \Big|_{v=0}.$$

Using

$$\Gamma(1-k+v+n) = \begin{cases} \Gamma(1-k+v+n) & n \geq k \\ \frac{\pi}{\sin \pi(k-v-n) \Gamma(k-n-v)} & n \leq k-1 \end{cases},$$

we have

$$\phi_k(x) = \sum_{n=0}^{k-1} \frac{(-1)^n x^n}{n!} \frac{\Gamma(k+n)}{\Gamma(k-n) \Gamma(n+1)} \times \\ \left(-2 \log x + 4 \frac{\Gamma'}{\Gamma}(n+1) - 2 \frac{\Gamma'}{\Gamma}(k+n) - 2 \frac{\Gamma'}{\Gamma}(k-n) \right) + \\ 2(-1)^k \sum_{n=k}^{\infty} \frac{\Gamma(n+k) \Gamma(n-k+1) x^n}{\Gamma(n+1) n!}.$$

Then the assertion follows by applying [19, Eq. 15.2.4]. \square

Lemma 5.4. *The function $\phi_k(x)$ satisfies the differential equation*

$$(5.5) \quad (x-x^2)\phi_k''(x) + (1-2x)\phi_k'(x) + k(k-1)\phi_k(x) = 0.$$

Proof. Note that ${}_2F_1(k, 1-k, 1; x)$ is a solution of equation (5.5). Using (5.4) we can write

$$\phi_k(x) = -2\alpha_1 - 2\alpha_2 + 2(-1)^k \alpha_3,$$

where

$$\alpha_1 := {}_2F_1(k, 1-k, 1; x) \log x,$$

$$\alpha_2 := \sum_{n=0}^{k-1} A(n)B(n)x^n, \quad \alpha_3 := \sum_{n=k}^{\infty} C(n)x^n.$$

Coefficients $A(n)$, $B(n)$, $C(n)$ are defined by

$$\begin{aligned} A(n) &:= (-1)^n \frac{\Gamma(k+n)}{\Gamma^2(n+1)\Gamma(k-n)}, \\ B(n) &:= \frac{\Gamma'}{\Gamma}(n+k) + \frac{\Gamma'}{\Gamma}(n-k) - 2\frac{\Gamma'}{\Gamma}(n+1), \\ C(n) &:= \frac{\Gamma(n+k)\Gamma(n-k+1)}{\Gamma^2(n+1)}. \end{aligned}$$

They satisfy recurrence relations

$$\begin{aligned} A(n+1) &= -\frac{(k+n)(k-n-1)}{(n+1)^2} A(n), \\ B(n+1) &= B(n) + \frac{1}{k+n} - \frac{1}{k-n-1} - \frac{2}{n+1}, \\ C(n+1) &= \frac{(n+k)(n-k+1)}{(n+1)^2} C(n). \end{aligned}$$

Let us denote

$$D(f) = (x - x^2)f'' + (1 - 2x)f' + k(k-1)f.$$

Using the recurrence relations above, we compute $D(\alpha_1)$, $D(\alpha_2)$, $D(\alpha_3)$ and prove the Lemma by showing that

$$D(\alpha_1) + D(\alpha_2) = (-1)^k D(\alpha_3).$$

□

Lemma 5.5. *Let $y = y(x)$ be a solution of differential equation*

$$(5.6) \quad A(x)y''(x) + B(x)y'(x) + C(x)y(x) = 0.$$

Then $z(x) = y(x)/\alpha(x)$ satisfies equation

$$(5.7) \quad A_1(x)z''(x) + B_1(x)z'(x) + C_1(x)z(x) = 0,$$

where

$$(5.8) \quad A_1(x) = A(x)\alpha(x), \quad B_1(x) = 2A(x)\alpha'(x) + B(x)\alpha(x),$$

$$(5.9) \quad C_1(x) = A(x)\alpha''(x) + B(x)\alpha'(x) + C(x)\alpha(x).$$

Corollary 5.6. *The function $Y(x) := \sqrt{x(1-x)}\phi_k(x)$ is a solution of differential equation*

$$(5.10) \quad Y''(x) + \left(\frac{1}{4x^2(1-x)^2} + \frac{k(k-1)}{x(1-x)} \right) Y(x) = 0.$$

Proof. Apply Lemma 5.5 with $\alpha(x) = 1/\sqrt{x(1-x)}$. \square

Lemma 5.7. *Assume that k is an even positive integer. Then*

$$(5.11) \quad {}_2F_1(k, 1-k, 1; 1/2) = 0,$$

$$(5.12) \quad \left. \frac{d}{dx} ({}_2F_1(k, 1-k, 1; x)) \right|_{x=1/2} = (-1)^{k/2} \frac{4\Gamma(1/2)\Gamma((k+1)/2)}{\pi\Gamma(k/2)}.$$

Proof. Using formula [19, Eq. 15.8.25] and Euler's reflection formula, we have

$$\begin{aligned} {}_2F_1(a, 1-a, 1; x) &= \frac{\Gamma(1/2)}{\pi} \frac{\Gamma(a/2) \sin(\pi a/2)}{\Gamma((a+1)/2)} \times \\ &\quad {}_2F_1\left(\frac{a}{2}, \frac{1-a}{2}, \frac{1}{2}; (1-2x)^2\right) - (1-2x) \frac{2\Gamma(1/2)}{\pi} \times \\ &\quad \frac{\Gamma((a+1)/2) \sin(\pi(a+1)/2)}{\Gamma(a/2)} {}_2F_1\left(\frac{a+1}{2}, 1-\frac{a}{2}, \frac{3}{2}; (1-2x)^2\right) \end{aligned}$$

for some complex variable a . Setting $a = k$ we obtain

$$\begin{aligned} {}_2F_1(k, 1-k, 1; x) &= -(-1)^{k/2} \frac{2\Gamma(1/2)\Gamma((k+1)/2)}{\pi\Gamma(k/2)} (1-2x) \times \\ &\quad {}_2F_1\left(\frac{k+1}{2}, 1-\frac{k}{2}, \frac{3}{2}; (1-2x)^2\right). \end{aligned}$$

Equation (5.11) follows by taking $x = 1/2$.

Equation (5.12) is obtained by differentiation of ${}_2F_1(k, 1-k, 1; x)$ with respect to x using the series representation

$$\begin{aligned} {}_2F_1\left(\frac{k+1}{2}, 1-\frac{k}{2}, \frac{3}{2}; (1-2x)^2\right) &= \\ &\quad \sum_{n=0}^{k/2-1} (-1)^n \binom{k/2-1}{n} \frac{\Gamma(k/2+1/2+n)\Gamma(3/2)}{\Gamma(k/2+1/2)\Gamma(3/2+n)} (1-2x)^{2n}. \end{aligned}$$

This representation is a consequence of [19, Eq. 15.2.4] for even positive integer k . \square

Lemma 5.8. *For even positive integer k one has*

$$(5.13) \quad \phi_k(1/2) = 2\sqrt{\pi}(-1)^{k/2} \frac{\Gamma(k/2)}{\Gamma((k+1)/2)}.$$

Proof. By formulas (5.2) and (5.11) we have

$$\phi_k(1/2) = -2 \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2 \frac{\partial}{\partial c} \right) {}_2F_1(a, b, c; 1/2) \Big|_{\substack{a=k \\ b=1-k \\ c=1}}.$$

Define two functions

$$G_1 := {}_2F_1(k + 2\epsilon, 1 - k, 1 + \epsilon; z),$$

$$G_2 := {}_2F_1(k - \epsilon, 1 - k + \epsilon, 1 + \epsilon; z).$$

By [19, Eq. 15.8.26]

$$G_1 = \frac{\Gamma(1/2)}{\pi} (1 - z)^\epsilon \frac{\Gamma(1 + \epsilon) \Gamma(k/2 - \epsilon)}{\Gamma(k/2 + 1/2)} \sin \frac{\pi}{2} (k - 2\epsilon) g_{1,1} +$$

$$\frac{\Gamma(-1/2)}{\pi} (1 - 2z)(1 - z)^\epsilon \frac{\Gamma(1 + \epsilon) \Gamma(k/2 + 1/2 - \epsilon)}{\Gamma(k/2)} \sin \frac{\pi}{2} (k + 1 - 2\epsilon) g_{1,2},$$

where

$$g_{1,1} = {}_2F_1(1/2 - k/2 + \epsilon, k/2, 1/2; (1 - 2z)^2),$$

$$g_{1,2} = {}_2F_1(1 - k/2 + \epsilon, k/2 + 1/2, 3/2; (1 - 2z)^2).$$

By [19, Eq. 15.8.25]

$$G_2 = -\frac{2(-1)^{k/2} \Gamma(1/2)}{\pi} (1 - 2z) \frac{\Gamma(1 + \epsilon) \Gamma(k/2 + 1/2)}{\Gamma(k/2 + \epsilon)} g_{2,2},$$

where

$$g_{2,2} = {}_2F_1(k/2 + 1/2 + \epsilon, 1 - k/2, 3/2; (1 - 2z)^2).$$

Differentiating G_1 and G_2 with respect to ϵ at the point $\epsilon = 0$ and summing the results, we have

$$\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} + 2 \frac{\partial}{\partial c} \right) {}_2F_1(a, b, c; z) \Big|_{\substack{a=k \\ b=1-k \\ c=1}} = -(1 - 2z) \log(1 - z) \times$$

$$(-1)^{k/2} {}_2F_1(k/2, 1/2 - k/2, 1/2; (1 - 2z)^2) \frac{2\Gamma(k/2 + 1/2) \Gamma(1/2)}{\pi \Gamma(k/2)} -$$

$$(-1)^{k/2} {}_2F_1(k/2, 1/2 - k/2, 1/2; (1 - 2z)^2) \frac{\Gamma(1/2) \Gamma(k/2)}{\Gamma(k/2 + 1/2)} -$$

$$\frac{\Gamma(k/2 + 1/2) \Gamma(1/2)}{\pi \Gamma(k/2)} (1 - 2z) \left({}_2F_1(k/2 + 1/2, 1 - k/2, 3/2; (1 - 2z)^2) \times \right.$$

$$(-1)^{k/2} \left(4 \frac{\Gamma'}{\Gamma}(1) - 2 \frac{\Gamma'}{\Gamma}(k/2 + 1/2) - 2 \frac{\Gamma'}{\Gamma}(k/2) \right) +$$

$$\left. 2(-1)^{k/2} \left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b} \right) {}_2F_1(a, b, 3/2; (1 - 2z)^2) \Big|_{\substack{a=(k+1)/2 \\ b=1-k/2}} \right).$$

Setting $z = 1/2$ we prove the Lemma. \square

5.2. Asymptotic approximation of ϕ_k . We apply the Liouville-Green method to find a uniform approximation of the function $\phi_k(x)$.

In Corollary 5.6 we showed that $y(x) = \sqrt{x(1-x)}\phi_k(x)$ is a solution of differential equation (5.10). This equation is a particular type of [3, Eq. 1.1] when $\alpha = 0$. Let

$$(5.14) \quad u := k - 1/2, \quad f(x) := -\frac{1}{x(1-x)},$$

$$(5.15) \quad g(x) := -\frac{1}{4x^2(1-x)^2} + \frac{1}{4x(1-x)}.$$

Then equation (5.10) can be written as

$$(5.16) \quad y''(x) = (u^2 f(x) + g(x))y(x).$$

Note that $x^2 g(x) \rightarrow -1/4$ as $x \rightarrow 0$. We would like to transform equation (5.16) into the following shape

$$(5.17) \quad \frac{d^2 Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2} + \frac{\psi(\xi)}{\xi} \right] Z = 0,$$

which corresponds to [3, Eq. 2.4, 2.6].

Let $\alpha(x)$, $\eta(x)$ be some suitable functions (to be chosen later). We make the change

$$(5.18) \quad Z(x) := \frac{y(x)}{\alpha(x)}$$

in equation (5.16) and apply Lemma 5.5. Then substitution

$$(5.19) \quad \xi := \int \eta(x) dx$$

gives

$$(5.20) \quad \alpha(x)\eta^2(x)\frac{d^2 Z}{d\xi^2} + (\alpha(x)\eta'(x) + 2\alpha'(x)\eta(x))\frac{dZ}{d\xi} + (\alpha''(x) - \alpha(x)(u^2 f(x) + g(x)))Z(\xi) = 0.$$

In order to obtain equation (5.17) we make the coefficient before $\frac{dZ}{d\xi}$ vanish by requiring

$$(5.21) \quad \alpha^2(x)\eta(x) = 1.$$

Next, we assume that $-\alpha^4(x)f(x) = 1/(4\xi)$. This implies

$$(5.22) \quad \xi = 4 \arcsin^2 \sqrt{x}, \quad \alpha(x) = \frac{(x - x^2)^{1/4}}{2(\arcsin \sqrt{x})^{1/2}}.$$

Using (5.21) and (5.22), equation (5.20) can be transformed into (5.17) with

$$(5.23) \quad \psi(\xi) := \frac{1}{16 \sin^2 \sqrt{\xi}} - \frac{1}{16\xi}.$$

Note that $\psi(\xi)$ is regular in the ξ -plane apart from poles at $\xi = (\pi m)^2$, $m \neq 0$. Since

$$\psi(\xi) = -\frac{1}{48} + O(\xi) \text{ as } \xi \rightarrow 0,$$

the function $\psi(\xi)$ is smooth on interval $[0, \delta]$ for any $0 < \delta < \pi^2$.

Removing the summand with $\psi(\xi)/\xi$ in equation (5.17), we have

$$(5.24) \quad \frac{d^2 Z}{d\xi^2} + \left[\frac{u^2}{4\xi} + \frac{1}{4\xi^2} \right] Z = 0.$$

The solutions of (5.24) are defined by

$$(5.25) \quad Z_C = \sqrt{\xi} C_0(u\sqrt{\xi}),$$

where C_i is either J or Y Bessel function of index i . Note that

$$(5.26) \quad \frac{d}{dz} C_0(z) = -C_1(z).$$

Therefore, following [18, Chapter 12] we are searching for a solution of differential equation (5.17) in the form

$$(5.27) \quad Z_C(\xi) = \sqrt{\xi} C_0(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{A(n; \xi)}{u^{2n}} - \frac{\xi}{u} C_1(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{B(n; \xi)}{u^{2n}}.$$

And our problem reduces to finding coefficients $A(n; \xi)$, $B(n; \xi)$. Let us denote

$$(5.28) \quad W(\xi) := \sqrt{\xi} C_0(u\sqrt{\xi}), \quad V(\xi) := \xi C_1(u\sqrt{\xi}).$$

These functions satisfy differential equations (see [6, Eq. 8.491(3)])

$$(5.29) \quad W''(\xi) + \left(\frac{u^2}{4\xi} + \frac{1}{4\xi^2} \right) W(x) = 0,$$

$$(5.30) \quad V''(\xi) - \frac{1}{\xi} V'(\xi) + \left(\frac{u^2}{4\xi} + \frac{3}{4\xi^2} \right) V(x) = 0.$$

Note that

$$(5.31) \quad W'(\xi) = \frac{1}{2\xi} W(\xi) - \frac{u}{2\xi} V(\xi),$$

$$(5.32) \quad V'(\xi) = \frac{1}{2\xi} V(\xi) + \frac{u}{2} W(\xi).$$

Then, substituting (5.27) in equation (5.17), we find

$$(5.33) \quad W(\xi) \sum_{n=0}^{\infty} \frac{C_n(\xi)}{u^{2n}} - V(\xi) \sum_{n=0}^{\infty} \frac{D_n(\xi)}{u^{2n-1}} = 0,$$

where

$$C_n(\xi) := A''(n; \xi) + \frac{1}{\xi} A'(n; \xi) - \frac{\psi(\xi)}{\xi} A(n; \xi) - B'(n; \xi) - \frac{B(n; \xi)}{2\xi},$$

$$D_n(\xi) := B''(n-1; \xi) + \frac{1}{\xi} B'(n-1; \xi) - \frac{\psi(\xi)}{\xi} B(n-1; \xi) + \frac{1}{\xi} A'(n; \xi).$$

Assuming that $C_n(\xi) = D_n(\xi) = 0$, one has

$$\sqrt{\xi}(\sqrt{\xi}B(n; \xi))' = \xi A''(n; \xi) + A'(n; \xi) - \psi(\xi)A(n; \xi),$$

$$A'(n; \xi) = -(\xi B'(n-1; \xi))' + \psi(\xi)B(n-1; \xi).$$

This yields the following recurrence relations

$$(5.34) \quad A(n; \xi) = -\xi B'(n-1; \xi) + \int_0^\xi \psi(x)B(n-1; x)dx + \lambda_n,$$

$$(5.35) \quad \sqrt{\xi}B(n; \xi) = \int_0^\xi \frac{1}{\sqrt{x}} (xA''(n; x) + A'(n; x) - \psi(x)A(n; x)) dx$$

for some real constants of integration λ_n (to be chosen later). Letting $A(0; \xi) = 1$, one has

$$(5.36) \quad B(0; \xi) = -\frac{1}{8\sqrt{\xi}} \left(\cot \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \right),$$

$$(5.37) \quad A(1; \xi) = \frac{1}{8} \left(\frac{1}{\xi} - \frac{\cot \sqrt{\xi}}{2\sqrt{\xi}} - \frac{1}{2\sin^2 \sqrt{\xi}} \right) - \frac{1}{128} \left(\cot \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \right)^2 + \lambda_1.$$

The next step is to apply [18, Theorem 4.1, p. 444] or [3, Theorem 1]. This allows approximating ϕ_k by a finite series plus the error term.

Theorem 5.9. *Let $\xi_2 = \pi^2/4$. For each value of u and each nonnegative integer N equation (5.17) has solutions $Z_Y(\xi)$, $Z_J(\xi)$ which are*

infinitely differentiable in ξ on interval $(0, \xi_2)$, and are given by

$$(5.38) \quad Z_Y(\xi) = \sqrt{\xi} Y_0(u\sqrt{\xi}) \sum_{n=0}^N \frac{A_Y(n; \xi)}{u^{2n}} - \frac{\xi}{u} Y_1(u\sqrt{\xi}) \sum_{n=0}^{N-1} \frac{B_Y(n; \xi)}{u^{2n}} + \epsilon_{2N+1,1}(u, \xi),$$

$$(5.39) \quad Z_J(\xi) = \sqrt{\xi} J_0(u\sqrt{\xi}) \sum_{n=0}^N \frac{A_J(n; \xi)}{u^{2n}} - \frac{\xi}{u} J_1(u\sqrt{\xi}) \sum_{n=0}^{N-1} \frac{B_J(n; \xi)}{u^{2n}} + \epsilon_{2N+1,2}(u, \xi),$$

where

$$(5.40) \quad \epsilon_{2N+1,1}(u, \xi) \ll \frac{\sqrt{\xi} |Y_0(u\sqrt{\xi})|}{u^{2N+1}} \sqrt{\xi_2 - \xi},$$

$$(5.41) \quad \epsilon_{2N+1,2}(u, \xi) \ll \frac{\sqrt{\xi} |J_0(u\sqrt{\xi})|}{u^{2N+1}} \min(\sqrt{\xi}, 1)$$

and coefficients $(A_Y(n; \xi), B_Y(n; \xi))$, $(A_J(n; \xi), B_J(n; \xi))$ satisfy recurrence relations (5.34)-(5.35).

Furthermore, there are $C_Y = C_Y(u)$ and $C_J = C_J(u)$ such that

$$(5.42) \quad \xi^{1/4} (\sin \sqrt{\xi})^{1/2} \phi_k \left(\sin^2 \frac{\sqrt{\xi}}{2} \right) = C_Y Z_Y(\xi) + C_J Z_J(\xi).$$

Our goal now is to determine explicitly C_Y and C_J .

As $\xi \rightarrow 0$, both ${}_2F_1(k, 1-k, 1; \sin^2 \sqrt{\xi}/2)$ and $Z_J(\xi)$ are recessive solutions. Thus there is a constant c_0 such that

$$(5.43) \quad \xi^{1/4} (\sin \sqrt{\xi})^{1/2} {}_2F_1(k, 1-k, 1; \sin^2 \sqrt{\xi}/2) = c_0 Z_J(\xi).$$

Note that

$$(5.44) \quad \lim_{\xi \rightarrow 0} {}_2F_1(k, 1-k, 1; \sin^2 \sqrt{\xi}/2) = 1.$$

By [19, Eq. 10.2.2] one has $J_0(x) = 1 + O(x^2)$ and $J_1(x) = x/2 + O(x^3)$. Therefore,

$$(5.45) \quad Z_J(\xi) = \sqrt{\xi} \sum_{n=0}^N \frac{A_J(n; \xi)}{u^{2n}} + O(\xi) \text{ as } \xi \rightarrow 0.$$

Choosing $A_J(n; \xi)$ such that $A_J(0; 0) = 1$ and $A_J(n; 0) = 0$ for $n \geq 1$ we find that $\lim_{\xi \rightarrow 0} Z_J(\xi) = \sqrt{\xi}$ and $c_0 = 1$.

In contrast, functions $\phi_k \left(\sin^2 \frac{\sqrt{\xi}}{2} \right)$ and $Z_Y(\xi)$ are dominant for any $0 < \xi < \xi_2$. To solve this problem, we apply the method described in [18, §12.5].

Differential equation (5.17) has two solutions $\phi_k(\xi)$ and ${}_2F_1(k, 1 - k, 1; \xi)$, which are linearly independent since

$$\phi_k(\xi) \sim \log \xi, \quad {}_2F_1(k, 1 - k, 1; \xi) \sim 1 \text{ as } \xi \rightarrow 0.$$

Therefore, Z_Y can be written as a linear combination

$$(5.46) \quad Z_Y(\xi) = \xi^{1/4} (\sin \sqrt{\xi})^{1/2} \times \left(\phi_k \left(\sin^2 \frac{\sqrt{\xi}}{2} \right) c_1 + {}_2F_1 \left(k, 1 - k, 1; \sin^2 \frac{\sqrt{\xi}}{2} \right) c_2 \right)$$

for some constants c_1, c_2 .

Substituting (5.43) with $c_0 = 1$ into (5.46) we have

$$(5.47) \quad \xi^{1/4} (\sin \sqrt{\xi})^{1/2} \phi_k \left(\sin^2 \frac{\sqrt{\xi}}{2} \right) = \frac{1}{c_1} Z_Y(\xi) - \frac{c_2}{c_1} Z_J(\xi)$$

provided that $c_1 \neq 0$.

In order to determine constants c_1, c_2 we compute $Z_Y(\xi)$ and its derivative at $\xi_2 = \pi^2/4$. Applying Lemmas 5.7, 5.8 one has

$$(5.48) \quad Z_Y(\xi_2) = \xi_2^{1/4} c_1 \phi_k(1/2),$$

$$(5.49) \quad Z'_Y(\xi_2) = \frac{Z_Y(\xi_2)}{4\xi_2} + \frac{c_2}{4\xi_2^{1/4}} \frac{\partial}{\partial x} {}_2F_1(k, 1 - k, 1; x) \Big|_{x=1/2}.$$

We find

$$(5.50) \quad c_1 = (-1)^{k/2} \frac{\Gamma(k/2 + 1/2)}{2\Gamma(1/2)\Gamma(k/2)} \frac{Z_Y(\xi_2)}{\xi_2^{1/4}},$$

$$(5.51) \quad c_2 = (-1)^{k/2} \frac{\pi\Gamma(k/2)}{\Gamma(1/2)\Gamma(k/2 + 1/2)} \xi_2^{1/4} \left(Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{4\xi_2} \right).$$

The final step is to compute $Z_Y(\xi_2)$ and $Z'_Y(\xi_2)$.

Lemma 5.10. *For $\xi_2 = \pi^2/4$ the following asymptotic formulas hold*

$$(5.52) \quad Z_Y(\xi_2) = \frac{(-1)^{k/2+1}}{\sqrt{u}} \left(1 + \frac{1}{u^2} (\lambda_1 - 1/16) + O(u^{-4}) \right).$$

$$(5.53) \quad Z'_Y(\xi_2) = \frac{(-1)^{k/2+1}}{\sqrt{u}\pi^2} \left(1 + \frac{1}{u^2} \left[\frac{5\lambda_1}{4} - \frac{5}{64} - \frac{405}{128\pi^2} \right] \right) + O(u^{-5/2}).$$

Proof. By [3, Theorem 1] we have that

$$\epsilon_{2N+1,1}(u; \xi_2) = 0 \text{ and } \left. \frac{\partial}{\partial \xi} \epsilon_{2N+1,1}(u; \xi) \right|_{\xi=\xi_2} = 0.$$

Therefore,

$$Z_Y(\xi_2) = \sqrt{\xi_2} Y_0(u\sqrt{\xi_2}) \sum_{n=0}^N \frac{A_Y(n; \xi_2)}{u^{2n}} - \frac{\xi_2}{u} Y_1(u\sqrt{\xi_2}) \sum_{n=0}^{N-1} \frac{B_Y(n; \xi_2)}{u^{2n}}.$$

Using relations (5.31) and (5.32), we obtain

$$\begin{aligned} Z'_Y(\xi_2) = \sqrt{\xi_2} Y_0(u\sqrt{\xi_2}) & \left(\frac{1}{2\xi_2} \sum_{n=0}^N \frac{A_Y(n; \xi_2)}{u^{2n}} + \sum_{n=0}^N \frac{A'_Y(n; \xi_2)}{u^{2n}} - \right. \\ & \left. \frac{1}{2} \sum_{n=0}^{N-1} \frac{B_Y(n; \xi_2)}{u^{2n}} \right) - \xi_2 Y_1(u\sqrt{\xi_2}) \left(\frac{u}{2\xi_2} \sum_{n=0}^N \frac{A_Y(n; \xi_2)}{u^{2n}} + \right. \\ & \left. \frac{1}{2\xi_2 u} \sum_{n=0}^{N-1} \frac{B_Y(n; \xi_2)}{u^{2n}} + \frac{1}{u} \sum_{n=0}^{N-1} \frac{B'_Y(n; \xi_2)}{u^{2n}} \right). \end{aligned}$$

For our purposes it is sufficient to take $N = 1$. Applying [19, Eq. 10.17.1, 10.17.4] and [6, Eq. 8.451(1,7,8)] we can write Hankel's expansions for Bessel functions

$$\sqrt{\xi_2} Y_0(u\sqrt{\xi_2}) = \frac{(-1)^{k/2+1}}{\sqrt{u}} \sum_{j=0}^{\infty} (-1)^j \frac{a_{2j}(0)}{(\pi u/2)^{2j}},$$

$$\xi_2 Y_1(u\sqrt{\xi_2}) = \frac{\pi}{2} \frac{(-1)^{k/2+1}}{\sqrt{u}} \sum_{j=0}^{\infty} (-1)^j \frac{a_{2j+1}(1)}{(\pi u/2)^{2j+1}},$$

where

$$a_j(v) = \frac{\Gamma(v + j + 1/2)}{2^j j! \Gamma(v - j + 1/2)}.$$

Thus

$$\sqrt{\xi_2} Y_0(u\sqrt{\xi_2}) = \frac{(-1)^{k/2+1}}{\sqrt{u}} \left(1 - \frac{a_2(0)}{(\pi u/2)^2} + O(u^{-4}) \right),$$

$$\xi_2 Y_1(u\sqrt{\xi_2}) = \frac{\pi}{2} \frac{(-1)^{k/2+1}}{\sqrt{u}} \left(\frac{a_1(1)}{\pi u/2} + O(u^{-3}) \right).$$

Consequently, taking $N = 1$

$$Z_Y(\xi_2) = \frac{(-1)^{k/2+1}}{\sqrt{u}} \left(A_Y(0; \xi_2) + \frac{1}{u^2} \left(A_Y(1; \xi_2) - \frac{A_Y(0; \xi_2)a_2(0)}{(\pi/2)^2} - B_Y(0; \xi_2)a_1(1) \right) + O(u^{-4}) \right).$$

According to (5.36) and (5.37), one has

$$A_Y(0; \xi_2) = 1, \quad A_Y(1; \xi_2) = -\frac{1}{16} + \frac{15}{32\pi^2} + \lambda_1, \quad B_Y(0; \xi_2) = \frac{1}{2\pi^2}.$$

Substituting $a_2(0) = 9/128$, $a_1(1) = 3/8$, we prove equation (5.52).

Next we find an asymptotic expansion for the derivative of $Z_Y(\xi)$ at $\xi = \xi_2 = \pi^2/4$. From recurrence relations (5.34) and (5.35) it follows that

$$A'_Y(1; \xi_2) = -\frac{15}{8\pi^4} + \frac{3}{32\pi^2}.$$

Taking $N = 1$, using Hankel's expansions for Bessel functions and computing $a_3(1) = -105/1024$, we obtain formula (5.53). \square

Lemma 5.11. *For $\lambda_1 = \frac{1}{16} + \frac{405}{32\pi^2}$ one has*

$$(5.54) \quad C_J = -\pi^2 \frac{\Gamma^2(k/2)}{\Gamma^2(k/2 + 1/2)} \frac{1}{Z_Y(\xi_2)} \left(Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{\pi^2} \right),$$

$$(5.55) \quad C_J = O(k^{-5}).$$

Proof. Note that with our choice of λ_1 the value of c_1 is non-zero. Therefore, by (5.42), (5.47), (5.50), (5.51) we have

$$C_J = -\frac{c_2}{c_1} = -\pi^2 \frac{\Gamma^2(k/2)}{\Gamma^2(k/2 + 1/2)} \frac{1}{Z_Y(\xi_2)} \left(Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{\pi^2} \right).$$

By Lemma 5.10

$$Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{\pi^2} = \frac{(-1)^{k/2+1}}{u^{5/2}} \left(\frac{\lambda_1}{4\pi^2} - \frac{405}{128\pi^4} - \frac{1}{64\pi^2} \right) + O(u^{-9/2}).$$

We choose λ_1 such that the first summand in the formula above is zero, so that

$$Z'_Y(\xi_2) - \frac{Z_Y(\xi_2)}{\pi^2} = O(u^{-9/2}).$$

Stirling's formula and [19, Eq. 5.5.5] yield

$$\frac{\Gamma^2(k/2)}{\Gamma^2(k/2 + 1/2)} = \left(\frac{2^{k-1}\Gamma^2(k/2)}{\sqrt{\pi}\Gamma(k)} \right)^2 = O(k^{-1})$$

Combining all results we find that $C_J = O(u^{-5}) = O(k^{-5})$. \square

Lemma 5.12. *For $\lambda_1 = \frac{1}{16} + \frac{405}{32\pi^2}$, $u = k - 1/2$ one has*

$$(5.56) \quad C_Y = (-1)^{k/2} \frac{2\Gamma(1/2)\Gamma(k/2)}{\Gamma(k/2 + 1/2)} \frac{\xi_2^{1/4}}{Z_Y(\xi_2)}.$$

For any $n \geq 1$ there exist constants d_1, d_2, \dots, d_n such that

$$(5.57) \quad C_Y = -2\pi \left(1 + \frac{d_1}{u} + \frac{d_2}{u^2} + \dots \frac{d_n}{u^n} + O(u^{-n-1}) \right).$$

Proof. Applying formulas (5.42), (5.47), (5.50), we prove (5.56). Asymptotics (5.57) follows by equation (5.52), Stirling's formula and [19, Eq. 5.5.5]. \square

Finally, we deduce the main result of this subsection.

Theorem 5.13. *Let $\xi_2 = \pi^2/4$. Then for any $\xi \in (0, \xi_2)$ the equality (5.42) holds with Z_Y, Z_J, C_Y, C_J given by (5.38), (5.39), (5.56), (5.54), respectively.*

5.3. Asymptotic approximation of ψ_k and Φ_k . In this section we estimate the functions defined by (4.9) and (4.10), namely

$$(5.58) \quad \Phi_k(x) = \Phi_k(x; 0; 0) = 2 \frac{\Gamma^2(k)}{\Gamma(2k)} x^k {}_2F_1(k, k, 2k; x),$$

$$(5.59) \quad \psi_k(x) = \psi_k(x; 0; 0) = 2 \frac{\Gamma^2(k)}{\Gamma(2k)} x^k {}_2F_1(k, k, 2k; -x).$$

By [2, Eq. 1-3, p. 105]

$$\psi_k(x) = \frac{2\Gamma^2(k)}{\Gamma(2k)} \left(\frac{x}{1+x} \right)^k {}_2F_1 \left(k, k, 2k; \frac{x}{1+x} \right) = \Phi_k \left(\frac{x}{1+x} \right).$$

Hence

$$(5.60) \quad \psi_k \left(\frac{l}{n} \right) = \Phi_k \left(\frac{l}{n+l} \right)$$

and it is sufficient to consider only function Φ_k .

Let $u := k - 1/2$ and

$$(5.61) \quad f(x) := \frac{1}{x^2(1-x)}, \quad g(x) := -\frac{1}{4x^2(1-x)^2} + \frac{1}{4x(1-x)}.$$

Lemma 5.14. *The function $y(x) = {}_2F_1(k, k, 2k; x)x^k\sqrt{1-x}$ is a solution of differential equation*

$$(5.62) \quad y''(x) - (u^2 f(x) + g(x))y(x) = 0$$

Proof. The hypergeometric function $F(x) = {}_2F_1(k, k, 2k; x)$ satisfies equation

$$x(1-x)F''(x) + (2k - (2k+1)x)F'(x) - k^2F(x) = 0.$$

Applying Lemma 5.5 with

$$\alpha(x) = x^{-k}(1-x)^{-1/2}$$

we have

$$y''(x) + \left(\frac{1 - (2k-1)^2}{4x^2} + \frac{1}{4(1-x)^2} + \frac{1 - (2k-1)^2}{4x(1-x)} \right) y(x) = 0.$$

The assertion follows by rearranging the expression in brackets. \square

Note that equation (5.62) differs from (5.16) by the sign in $f(x)$. According to [18, Chapter 12] this means that one has to choose I and K Bessel functions (instead of Y and J) as approximation functions. Consequently, we transform (5.62) to the type

$$(5.63) \quad \frac{d^2 Z}{d\xi^2} + \left[-\frac{u^2}{4\xi} + \frac{1}{4\xi^2} - \frac{\psi(\xi)}{\xi} \right] Z = 0,$$

where

$$(5.64) \quad \psi(\xi) = \frac{1}{16} \left(\frac{1}{\xi} - \frac{1}{\sinh^2 \sqrt{\xi}} \right).$$

This can be done similarly to the previous case by making the change

$$(5.65) \quad Z(x) := \frac{y(x)}{\alpha(x)}, \quad \alpha(x) := \frac{(x^2 - x^3)^{1/4}}{2(\operatorname{artanh} \sqrt{1-x})^{1/2}}$$

and the substitution

$$(5.66) \quad \xi := 4 \operatorname{artanh}^2 \sqrt{1-x}.$$

Note that as $\xi \rightarrow 0$ the function $\psi(\xi)$ is analytic. Removing the term with $\psi(\xi)/\xi$ in (5.63) we obtain

$$(5.67) \quad Z'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi} \right) Z = 0.$$

The solutions of this equation are given by (see [19, Eq. 10.13.2])

$$(5.68) \quad Z_L = \sqrt{\xi} L_0(u\sqrt{\xi}),$$

where L_0 is either K_0 or I_0 Bessel function. In general,

$$(5.69) \quad L_v := \begin{cases} I_v \\ e^{\pi i v} K_v \end{cases}.$$

Therefore, according to [18, Eq. 2.09, Chapter 12] the solution of differential equation (5.63) can be found in the form

$$(5.70) \quad Z_L = \sqrt{\xi} L_0(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{A(n; \xi)}{u^{2n}} + \frac{\xi}{u} L_1(u\sqrt{\xi}) \sum_{n=0}^{\infty} \frac{B(n; \xi)}{u^{2n}}.$$

To determine coefficients $A(n; \xi)$ and $B(n; \xi)$ we introduce two functions

$$(5.71) \quad W(\xi) := \sqrt{\xi} L_0(u\sqrt{\xi}), \quad V(\xi) := \xi L_1(u\sqrt{\xi}),$$

which satisfy differential equations (see [19, Eq. 10.13.2, 10.13.5, 10.36])

$$(5.72) \quad W'' + \left(-\frac{u^2}{4\xi} + \frac{1}{4\xi^2} \right) W = 0,$$

$$(5.73) \quad V'' - \frac{1}{\xi} V' + \left(-\frac{u^2}{4\xi} + \frac{3}{4\xi^2} \right) V = 0.$$

Furthermore, using formulas 10.29.2 and 10.29.3 we prove that

$$(5.74) \quad V' = \frac{1}{2\xi} V + \frac{u}{2} W, \quad W' = \frac{1}{2\xi} W + \frac{u}{2\xi} V.$$

Substituting equation (5.70) into (5.63) one has

$$(5.75) \quad W(\xi) \sum_{n=0}^{\infty} \frac{C(n; \xi)}{u^{2n}} + V(\xi) \sum_{n=0}^{\infty} \frac{D(n; \xi)}{u^{2n+1}} = 0,$$

where

$$C(n; \xi) = A''(n; \xi) + \frac{A'(n; \xi)}{\xi} - \frac{\psi(\xi)}{\xi} A(n; \xi) + B'(n; \xi) + \frac{B(n; \xi)}{2\xi},$$

$$D(n; \xi) = B''(n-1; \xi) + \frac{B'(n-1; \xi)}{\xi} - \frac{\psi(\xi)}{\xi} B(n-1; \xi) + \frac{A'(n; \xi)}{\xi}.$$

Setting $C(n; \xi) = D(n; \xi) = 0$ we find recurrence relations

$$(5.76) \quad \sqrt{\xi} B(n; \xi) = -\sqrt{\xi} A'(n; \xi) + \int_0^\xi \left(\psi(x) A(n; x) - \frac{1}{2} A'(n; x) \right) \frac{dx}{\sqrt{x}},$$

$$(5.77) \quad A(n; \xi) = -\xi B'(n-1; \xi) + \int_0^\xi \psi(x) B(n-1; x) dx + \lambda_n$$

for some real constants of integration λ_n .

Let $A(0; \xi) = 1$. Then

$$(5.78) \quad B(0; \xi) = \frac{1}{8} \left(\frac{\coth \sqrt{\xi}}{\sqrt{\xi}} - \frac{1}{\xi} \right),$$

$$(5.79) \quad A(1; \xi) = -\frac{1}{8} \left(\frac{1}{\xi} - \frac{\coth \sqrt{\xi}}{2\sqrt{\xi}} - \frac{1}{2 \sinh^2 \sqrt{\xi}} \right) + \frac{1}{128} \left(\coth \sqrt{\xi} - \frac{1}{\sqrt{\xi}} \right)^2 + \lambda_1.$$

Note that

$$(5.80) \quad B(0; \xi) = \frac{1}{24} + O(\xi), \quad A(1; \xi) = \lambda_1 + O(\xi) \text{ as } \xi \rightarrow 0,$$

$$(5.81) \quad \lim_{\xi \rightarrow \infty} \sqrt{\xi} B(0; \xi) = \frac{1}{8}, \quad \lim_{\xi \rightarrow \infty} A(1; \xi) = \frac{1}{128} + \lambda_1.$$

The expression for $B(1; \xi)$ is quite complicated, so we do not write it explicitly. The only thing we need to show is that

$$\text{Var}_{\xi, \infty}(\sqrt{x}B(1; x)) = \int_{\xi}^{\infty} |(\sqrt{x}B(1; x))'| dx$$

is bounded. Using recurrence relation (5.76), we find

$$(5.82) \quad (\sqrt{x}B(1; x))' = -\frac{1}{\sqrt{x}} (xA''(1; x) + A'(1; x) - \psi(x)A(1; x)).$$

Therefore,

$$(5.83) \quad (\sqrt{x}B(1; x))' = O(x^{-1/2}) \text{ as } x \rightarrow 0$$

and

$$(5.84) \quad (\sqrt{x}B(1; x))' = O(x^{-2}) \text{ as } x \rightarrow \infty,$$

as required.

Note that

$$\psi^{(s)}(\xi) = O\left(\frac{1}{|\xi|^{s+1}}\right).$$

Hence for $n > 1$ the variation

$$\text{Var}_{\xi, \infty}(\sqrt{x}B(n; x)) = \int_{\xi}^{\infty} |(\sqrt{x}B(n; x))'| dx$$

converges by [18, Exercise 4.2, p.445].

Theorem 5.15. *For each value of u and each nonnegative integer N equation (5.63) has solution $Z_K(\xi)$ which is infinitely differentiable in*

ξ on interval $(0, \infty)$ and is given by

$$(5.85) \quad Z_K(\xi) = \sqrt{\xi} K_0(u\sqrt{\xi}) \sum_{n=0}^N \frac{A_K(n; \xi)}{u^{2n}} - \frac{\xi}{u} K_1(u\sqrt{\xi}) \sum_{n=0}^{N-1} \frac{B_K(n; \xi)}{u^{2n}} + \epsilon_{2N+1,3}(u, \xi),$$

where

$$(5.86) \quad |\epsilon_{2N+1,3}(u, \xi)| \leq \frac{\sqrt{\xi} K_0(u\sqrt{\xi})}{u^{2N+1}} \times V_{\xi, \infty}(\sqrt{\xi} B_K(N; \xi)) \exp \left(\frac{1}{u} V_{\xi, \infty}(\sqrt{\xi} B_K(0; \xi)) \right).$$

In particular, for $N = 1$

$$(5.87) \quad \epsilon_{3,3}(u, \xi) \ll \frac{\sqrt{\xi} K_0(u\sqrt{\xi})}{u^3} \min \left(\sqrt{\xi}, \frac{1}{\xi} \right).$$

The solution of differential equation (5.63)

$$(5.88) \quad Z(\xi) = \left(\Phi_k(x) \frac{\sqrt{1-x}}{\alpha(x)} \right) \Big|_{x=1/\cosh^2 \sqrt{\xi}/2} = \Phi_k \left(\frac{1}{\cosh^2 \sqrt{\xi}/2} \right) (\xi \sinh^2 \sqrt{\xi})^{1/4}$$

is recessive as $\xi \rightarrow \infty$. Another recessive solution is $Z_K(\xi)$ defined by (5.70) with $L_v = e^{\pi i v} K_v$. Therefore, there exists $C_K = C_K(u)$ such that

$$(5.89) \quad \Phi_k \left(\frac{1}{\cosh^2 \sqrt{\xi}/2} \right) (\xi \sinh^2 \sqrt{\xi})^{1/4} = C_K Z_K(\xi).$$

Computing the limit as $\xi \rightarrow \infty$ of the left and right- hand sides of equation (5.89), we find

$$(5.90) \quad C_K = 2 \frac{\Gamma^2(k)}{\Gamma(2k)} \frac{2^{2k} \sqrt{u}}{\sqrt{\pi}} \left[\sum_{n=0}^N \frac{a_n}{u^{2n}} - \sum_{n=0}^{N-1} \frac{b_n}{u^{2n+1}} \right]^{-1},$$

where

$$(5.91) \quad a_n = \lim_{\xi \rightarrow \infty} A(n; \xi), \quad b_n = \lim_{\xi \rightarrow \infty} B(n; \xi) \sqrt{\xi},$$

$$(5.92) \quad a_0 = 1, \quad a_1 = \frac{1}{128} + \lambda_1, \quad b_0 = \frac{1}{8}.$$

Since

$$\frac{\Gamma^2(k)}{\Gamma(2k)} = \frac{2\sqrt{\pi}}{\sqrt{k}2^{2k}}(1 + O(k^{-1})),$$

we have

$$(5.93) \quad C_K = 4 + O(k^{-1}).$$

To sum up, we proved the following result.

Theorem 5.16. *For $\xi \in (0, \infty)$ the equality (5.89) holds with Z_K, C_K given by (5.85), (5.90), respectively.*

6. ERROR TERMS FOR AN INDIVIDUAL WEIGHT

Lemma 6.1. *One has*

$$(6.1) \quad M_2(l; 0, 0) = \sum_{f \in H_{2k}(1)}^h \lambda_f(l) L_f^2(1/2) = (1 + (-1)^k) \times \\ \left(\frac{\tau(l)}{\sqrt{l}} \left[2 \frac{\Gamma'}{\Gamma}(k) - \log l - 2 \log(2\pi) + 2\gamma \right] + \right. \\ \left. \frac{1}{2\sqrt{l}} \sum_{n=1}^{l-1} \tau(n) \tau(l-n) \phi_k \left(\frac{n}{l} \right) + \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n) \tau(n+l) \Phi_k \left(\frac{l}{n+l} \right) \right).$$

Proof. The assertion follows from Theorem 4.2 by computing the limit as $u \rightarrow 0, v \rightarrow 0$. The error terms can be simplified as follows. First, by equation (5.3)

$$\sum_{n=1}^{l-1} \tau(n) \tau(n-l) \phi_k \left(\frac{n}{l} \right) = (-1)^k \sum_{n=1}^{l-1} \tau(n) \tau(l-n) \phi_k \left(\frac{n}{l} \right).$$

Second, by equation (5.60)

$$\psi_k \left(\frac{l}{n} \right) = \Phi_k \left(\frac{l}{n+l} \right).$$

Therefore,

$$\frac{1}{\sqrt{l}} \sum_{n=l+1}^{\infty} \tau(n) \tau(n-l) \Phi_k \left(\frac{l}{n} \right) + \frac{(-1)^k}{\sqrt{l}} \sum_{n=0}^{\infty} \tau(n) \tau(n+l) \psi_k \left(\frac{l}{n} \right) = \\ \frac{1 + (-1)^k}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n) \tau(n+l) \Phi_k \left(\frac{l}{n+l} \right).$$

□

Lemma 6.2. *For any $\epsilon > 0$, $l \ll k^2$ one has*

$$(6.2) \quad \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n) \tau(n+l) \Phi_{2k} \left(\frac{l}{n+l} \right) \ll \exp \left(-\frac{ck}{\sqrt{l}} \right) \max \left(\frac{l^\epsilon}{l^{1/4} k^{1/2}}, \frac{l^{1/2+\epsilon}}{k^{3/2}} \right)$$

for some absolute constant $c > 0$.

Proof. Consider

$$\begin{aligned} \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n) \tau(n+l) \Phi_{2k} \left(\frac{l}{n+l} \right) &\ll \frac{l^\epsilon}{\sqrt{l}} \sum_{n=1}^{\infty} n^\epsilon \left| \Phi_{2k} \left(\frac{l}{n+l} \right) \right| \ll \\ &\frac{l^\epsilon}{\sqrt{l}} \int_1^\infty x^\epsilon \left| \Phi_{2k} \left(\frac{l}{x+l} \right) \right| dx + O \left(\frac{l^\epsilon}{\sqrt{l}} \Phi_{2k} \left(\frac{l}{1+l} \right) \right). \end{aligned}$$

First, we estimate the integral above. Let

$$x := l \sinh^2 \frac{\sqrt{\xi}}{2}, \quad a := 2 \operatorname{arcsinh} \frac{1}{\sqrt{l}}.$$

Then

$$I := \frac{l^\epsilon}{\sqrt{l}} l \int_{a^2}^\infty \sinh^\epsilon \frac{\sqrt{\xi}}{2} \left| \Phi_{2k} \left(\frac{1}{\cosh^2 \frac{\sqrt{\xi}}{2}} \right) \right| \frac{\sinh \sqrt{\xi}}{\sqrt{\xi}} d\xi.$$

Applying the Liouville-Green method (see equation (5.89)) the integral is equal to

$$I = l^{1/2+\epsilon} C_K \int_{a^2}^\infty \sinh^\epsilon \frac{\sqrt{\xi}}{2} |Z_K(\xi)| \frac{(\sinh \sqrt{\xi})^{1/2}}{\xi^{3/4}} d\xi,$$

where C_K satisfies asymptotic formula (5.93). Note that

$$\sqrt{\xi} \geq 2 \operatorname{arcsinh} l^{-1/2} \gg 1/\sqrt{l}.$$

Thus

$$(2k-1/2)\sqrt{\xi} \gg k/\sqrt{l} \gg 1 \text{ for any } l \ll k^2.$$

Applying Theorem 5.15 with $N=0$ we estimate

$$Z_K(\xi) \ll \sqrt{\xi} K_0((2k-1/2)\sqrt{\xi}).$$

The last inequality and [19, Eq. 10.40.2] yield

$$\begin{aligned} I &\ll l^{1/2+\epsilon} \int_{a^2}^\infty \left| K_0((2k-1/2)\sqrt{\xi}) \right| \frac{(\sinh \sqrt{\xi})^{1/2+\epsilon}}{\sqrt{\xi}} d\xi \ll \\ &\frac{l^{1/2+\epsilon}}{\sqrt{2k-1/2}} \int_{a^2}^\infty \exp(-(2k-1/2)\sqrt{\xi}) \frac{(\sinh \sqrt{\xi})^{1/2+\epsilon}}{\sqrt{\xi}} d\xi. \end{aligned}$$

Making the change of variables $\xi = x^2$ and splitting the integral into two parts

$$\begin{aligned} I &\ll \frac{l^{1/2+\epsilon}}{\sqrt{k}} \int_a^\infty \exp(-(2k-1/2)x)(\sinh x)^{1/2+\epsilon} dx \ll \\ &\quad \frac{l^{1/2+\epsilon}}{\sqrt{k}} \left(\int_a^1 \exp(-(2k-1/2)x)x^{1/2+\epsilon} dx + \right. \\ &\quad \left. \int_1^\infty \exp(-(2k-1/2)x) \exp(x(1/2+\epsilon)) dx \right) \ll \frac{l^{1/2+\epsilon}}{k^{3/2}} \exp\left(-\frac{k}{\sqrt{l}}\right). \end{aligned}$$

Now we estimate the second term

$$E := \frac{l^\epsilon}{\sqrt{l}} \Phi_{2k} \left(\frac{l}{n+l} \right).$$

Let

$$\xi_0 := \left(2 \operatorname{artanh} \frac{1}{\sqrt{l+1}} \right)^2.$$

Then by equation (5.89)

$$E = \frac{l^\epsilon}{\sqrt{l}} C_K \frac{Z_K(\xi_0)}{(\xi_0 \sinh^2 \sqrt{\xi_0})^{1/4}}.$$

Note that $\xi_0 \sim l^{-1}$. Then for $l \ll k^2$ one has

$$Z_K(\xi_0) \ll \sqrt{\xi_0} K_0((2k-1/2)\sqrt{\xi_0}) \ll \frac{\xi_0^{1/4}}{\sqrt{k}} \exp(-(2k-1/2)\sqrt{\xi_0}).$$

Finally, for some constant $c > 0$ independent of k

$$E \ll \frac{l^\epsilon}{\sqrt{l}} \frac{\exp(-(2k-1/2)\sqrt{\xi_0})}{\sqrt{k}(\sinh(\sqrt{\xi_0}))^{1/2}} \ll \frac{l^\epsilon}{l^{1/4}k^{1/2}} \exp\left(-\frac{ck}{\sqrt{l}}\right).$$

□

Lemma 6.3. *For any $\epsilon > 0$, $l \ll k^2$*

$$(6.3) \quad \frac{1}{\sqrt{l}} \sum_{n=1}^{l-1} \tau(n)\tau(n-l)\phi_{2k}\left(\frac{n}{l}\right) \ll \frac{l^{1/2+\epsilon}}{\sqrt{k}}.$$

Proof. We apply equation (5.42) with $n/l =: \sin^2 \frac{\sqrt{\xi}}{2}$, so that

$$\phi_{2k}\left(\frac{n}{l}\right) = \frac{C_Y Z_Y(\xi) + C_J Z_J(\xi)}{(2 \arcsin \sqrt{n/l})^{1/2} (2n/l)^{1/4} (1-n/l)^{1/4}}.$$

Using property (5.3) we find

$$\frac{1}{\sqrt{l}} \sum_{n=1}^{l-1} \tau(n)\tau(n-l)\phi_{2k}\left(\frac{n}{l}\right) \ll \frac{l^\epsilon}{\sqrt{l}} \sum_{n \leq l/2} \left| \phi_{2k}\left(\frac{n}{l}\right) \right|.$$

Thus $n/l \leq 1/2$ and $\xi \leq \pi^2/4$. Since $n/l \geq 1/l$ one has $\xi \gg 1/l$. Hence

$$(2k - 1/2)\sqrt{\xi} \gg k/\sqrt{l} \gg 1 \text{ when } l \ll k^2.$$

Applying Theorem 5.9 with $N = 0$, we find

$$C_Y Z_Y(\xi) \ll \sqrt{\xi} Y_0((2k - 1/2)\sqrt{\xi}) \ll \frac{\xi^{1/4}}{k^{1/2}},$$

$$C_J Z_J(\xi) \ll \frac{\xi^{1/4}}{k^{11/2}}$$

since $C_J = O(k^{-5})$ by Lemma 5.11. Consequently,

$$\phi_{2k}\left(\frac{n}{l}\right) \ll \frac{1}{\sqrt{k}(n/l)^{1/4}(1 - n/l)^{1/4}}.$$

Finally,

$$\frac{l^\epsilon}{\sqrt{l}} \sum_{n \leq l/2} \left| \phi_{2k}\left(\frac{n}{l}\right) \right| \ll \frac{l^\epsilon}{\sqrt{l}} \sum_{n \leq l/2} \frac{(l/n)^{1/4}}{\sqrt{k}} \ll \frac{l^{1/2+\epsilon}}{\sqrt{k}}.$$

□

Combining all results together we prove the asymptotic formula.

Theorem 6.4. *For any $\epsilon > 0$, $l \ll k^2$, $k \equiv 0 \pmod{2}$ one has*

$$(6.4) \quad M_2(l) = \frac{2\tau(l)}{\sqrt{l}}(2 \log k - \log l - 2 \log 2\pi + 2\gamma) + O\left(\frac{l^{1/2+\epsilon}}{\sqrt{k}}\right).$$

7. ERROR TERMS ON AVERAGE

In this section we estimate the error terms averaged over k with a test function $h \in C_0^\infty(\mathbf{R}^+)$, which is non-negative, compactly supported on interval $[\theta_1, \theta_2]$ such that $\theta_2 > \theta_1 > 0$ and

$$(7.1) \quad \|h^{(n)}\|_1 \ll 1 \text{ for all } n \geq 0.$$

Let

$$(7.2) \quad H := \int_0^\infty h(y) dy, \quad H_1 := \int_0^\infty h(y) \log y dy.$$

We denote the averaged moments as follows

$$(7.3) \quad A_1(l) := \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h \lambda_f(l) L_f(1/2),$$

$$(7.4) \quad A_2(l) := \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h \lambda_f(l) L_f^2(1/2).$$

Lemma 7.1. *One has*

$$(7.5) \quad A_2(l) = \frac{2\tau(l)HK}{\sqrt{l}} \frac{1}{4} \left(2\log K - \log l - 2\log 8\pi + 2\gamma + 2\frac{H_1}{H} \right) + \\ O\left(\frac{1}{K\sqrt{l}}\right) + 2 \sum_k h\left(\frac{4k}{K}\right) \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n)\tau(n+l)\Phi_{2k}\left(\frac{l}{n+l}\right) + \\ \sum_k h\left(\frac{4k}{K}\right) \frac{1}{\sqrt{l}} \sum_{n=1}^{l-1} \tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right).$$

Proof. We average over k the result of Lemma 6.1. To compute the main term, we use [19, Eq. 5.11.2], namely

$$\frac{\Gamma'}{\Gamma}(k) \sim \log k - \frac{1}{2k} - \sum_{r=1}^{\infty} \frac{B_{2r}}{2rk^{2r}},$$

where B_{2r} are the Bernoulli numbers. Note that

$$\sum_k h\left(\frac{4k}{K}\right) \frac{1}{k} \ll \frac{1}{K}.$$

By Poisson's summation formula one has

$$2 \sum_k h\left(\frac{4k}{K}\right) \log k = 2 \sum_n \int_{-\infty}^{\infty} h\left(\frac{4x}{K}\right) \log x \exp(-2\pi inx) dx = \\ 2 \frac{K}{4} \sum_n \int_{-\infty}^{\infty} h(y) (\log y + \log K - \log 4) \exp\left(\frac{-2\pi inyK}{4}\right) dy.$$

If $n = 0$ this is equal to

$$2 \left(\frac{HK}{4} (\log K - \log 4) + \frac{H_1 K}{4} \right).$$

If $n \neq 0$ we integrate by parts $a \geq 2$ times and estimate the expression by its absolute value, obtaining

$$\int_{-\infty}^{\infty} \frac{\partial^a}{\partial y^a} \left(h(y) \log \frac{yK}{4} \right) \frac{1}{(nK)^a} dy \ll \frac{\log K}{(nK)^a}.$$

Similarly,

$$\sum_k h\left(\frac{4k}{K}\right) = \frac{HK}{4} + O\left(\frac{1}{K^a}\right).$$

□

Lemma 7.2. *For $l \ll K^2$, for any $A > 0$*

$$(7.6) \quad \sum_k h\left(\frac{4k}{K}\right) \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n) \tau(n+l) \Phi_{2k}\left(\frac{l}{n+l}\right) \ll l^{-A}.$$

Proof. According to subsection 5.3 the function Φ_k can be approximated by non-oscillatory K -Bessel functions. Therefore, it is sufficient to average the result of Lemma 6.2, i.e.

$$\begin{aligned} \sum_k h\left(\frac{4k}{K}\right) \frac{1}{\sqrt{l}} \sum_{n=1}^{\infty} \tau(n) \tau(n+l) \Phi_{2k}\left(\frac{l}{n+l}\right) &\ll \\ l^\epsilon \sum_k h\left(\frac{4k}{K}\right) \left(\frac{1}{l^{1/4} k^{1/2}} + \frac{l^{1/2}}{k^{3/2}} \right) \exp(-ck/\sqrt{l}) &\ll l^{-A}. \end{aligned}$$

□

Lemma 7.3. *For any $\epsilon > 0$, for any $a \geq 2$, for $l \ll K^2$ one has*

$$(7.7) \quad \sum_k h\left(\frac{4k}{K}\right) \frac{1}{\sqrt{l}} \sum_{n=1}^{l-1} \tau(n) \tau(l-n) \phi_{2k}\left(\frac{n}{l}\right) \ll \frac{l^{a/2-1/4+\epsilon}}{K^{a+1/2}} K + \frac{l^\epsilon}{\sqrt{l}K} K + \frac{l^{1/2+\epsilon}}{K^{7/2}} K.$$

Proof. By formula (5.3)

$$\phi_{2k}\left(\frac{n}{l}\right) = \phi_{2k}\left(\frac{l-n}{l}\right).$$

If l is odd

$$\sum_{n=1}^{l-1} \tau(n) \tau(l-n) \phi_{2k}\left(\frac{n}{l}\right) = 2 \sum_{n=1}^{l/2} \tau(n) \tau(l-n) \phi_{2k}\left(\frac{n}{l}\right),$$

and if l is even

$$\begin{aligned} \sum_{n=1}^{l-1} \tau(n) \tau(l-n) \phi_{2k}\left(\frac{n}{l}\right) &= 2 \sum_{n=1}^{l/2} \tau(n) \tau(l-n) \phi_{2k}\left(\frac{n}{l}\right) - \\ &\quad \phi_{2k}(1/2) \tau^2(l/2). \end{aligned}$$

Contribution of $\phi_{2k}(1/2)\tau^2(l/2)$ can be estimated using formula (5.13)

$$\begin{aligned} \frac{\tau^2(l/2)}{\sqrt{l}} \sum_k h\left(\frac{4k}{K}\right) \phi_{2k}(1/2) &= (-1)^k \frac{\tau^2(l/2)}{\sqrt{l}} \sum_k h\left(\frac{4k}{K}\right) \frac{2\sqrt{\pi}\Gamma(k)}{\Gamma(k+1/2)} \\ &\ll l^{-1/2+\epsilon} \sum_k h\left(\frac{4k}{K}\right) \frac{1}{\sqrt{k}} \ll l^{-1/2+\epsilon} \frac{1}{\sqrt{K}} \sum_k h\left(\frac{4k}{K}\right) \ll \frac{l^\epsilon}{l^{1/2}K^{1/2}}K. \end{aligned}$$

Next we estimate

$$\begin{aligned} \sum_k h\left(\frac{4k}{K}\right) \frac{1}{\sqrt{l}} \sum_{n=1}^{l/2} \tau(n)\tau(l-n)\phi_{2k}\left(\frac{n}{l}\right) &\ll \\ &\frac{l^\epsilon}{l^{1/2}} \sum_{n=1}^{l/2} \left| \sum_k h\left(\frac{4k}{K}\right) \phi_{2k}\left(\frac{n}{l}\right) \right|. \end{aligned}$$

Applying formula (5.42) we have

$$\phi_{2k}\left(\frac{n}{l}\right) = \frac{C_Y Z_Y(4 \arcsin^2 \sqrt{n/l}) + C_J Z_J(4 \arcsin^2 \sqrt{n/l})}{(2 \arcsin \sqrt{n/l})^{1/2} (2n/l)^{1/4} (1 - n/l)^{1/4}}.$$

Let $u := 2k - 1/2$, $\xi := 4 \arcsin^2 \sqrt{n/l}$. Then by equations (5.39), (5.55) we obtain

$$C_J Z_J(\xi) = O\left(\frac{\sqrt{\xi} J_0(u\sqrt{\xi})}{k^5}\right).$$

Since $u\sqrt{\xi} = (4k-1) \arcsin \sqrt{n/l} \gg K/\sqrt{l} \gg 1$ when $l \ll K^2$, one has

$$J_0(u\sqrt{\xi}) \ll \frac{1}{(u\sqrt{\xi})^{1/2}} \ll \frac{1}{\sqrt{k}(\arcsin \sqrt{n/l})^{1/2}}.$$

Therefore, contribution of the term with $C_J Z_J$ is bounded by

$$\begin{aligned} \frac{l^\epsilon}{\sqrt{l}} \sum_{n=1}^{l/2} \frac{(l/n)^{1/4}}{(\arcsin \sqrt{n/l})^{1/2}} \sum_k h\left(\frac{4k}{K}\right) \frac{\arcsin \sqrt{n/l} |J_0(u\sqrt{\xi})|}{k^5} &\ll \\ \frac{l^\epsilon}{\sqrt{l}} \sum_{n=1}^{l/2} \left(\frac{l}{n}\right)^{1/4} \sum_k h\left(\frac{4k}{K}\right) \frac{1}{k^{11/2}} &\ll \frac{l^{1/2+\epsilon}}{K^{11/2}}K. \end{aligned}$$

Now we estimate contribution of the term with $C_Y Z_Y$, namely

$$\frac{l^\epsilon}{\sqrt{l}} \sum_{n=1}^{l/2} \frac{(l/n)^{1/4}}{(\arcsin \sqrt{n/l})^{1/2}} \left| \sum_k h\left(\frac{4k}{K}\right) C_Y Z_Y(\xi) \right|.$$

We use the series representation (5.57) for C_Y with sufficiently large n . The main contribution comes from the first summand -2π .

The function $Z_Y(\xi)$ is defined by (5.38). By formula (5.40) the error term is majorized by

$$\epsilon_{3,1}(u, \xi) \ll \frac{\sqrt{\xi}|Y_0(u\sqrt{\xi})|}{u^3}$$

and, therefore, its contribution is bounded by

$$\frac{l^{1/2+\epsilon}}{K^{7/2}}K.$$

On the interval $(0, \pi^2/4)$ the functions $B_0(\xi)$, $A_1(\xi)$ are bounded, independent of k and non-oscillatory (see equations (5.36), (5.37)).

The Y -Bessel functions have oscillatory behavior. According to equation [6, Eq. 8.451(2)] one has

$$\begin{aligned} Y_v(z) = & \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi v}{2} - \frac{\pi}{4}\right) \times \\ & \left(\sum_{s=0}^{s_1-1} \frac{(-1)^s \Gamma(v+2s+1/2)}{(2z)^{2s} (2s)! \Gamma(v-2s+1/2)} + R_1 \right) + \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi v}{2} - \frac{\pi}{4}\right) \times \\ & \left(\sum_{s=0}^{s_1-1} \frac{(-1)^s \Gamma(v+2s+3/2)}{(2z)^{2s+1} (2s+1)! \Gamma(v-2s-1/2)} + R_2 \right), \end{aligned}$$

where $R_1 = O(z^{-2s_1})$, $R_2 = O(z^{-2s_1-1})$ by [6, Eq. 8.451(7,8)]. Since $u\sqrt{\xi} > 1$ the only difference between $Y_0(u\sqrt{\xi})$ and $Y_1(u\sqrt{\xi})$ is the shift on $\pi/2$ in the oscillating multiples. Thus one can consider only $Y_0(u\sqrt{\xi})$.

Contribution of R_1 , R_2 is majorized by

$$\frac{l^\epsilon}{\sqrt{l}} \sum_{n=1}^{l/2} \sum_k \frac{h(4k/K)}{(k\sqrt{n/l})^{2s_1+1/2}} \ll \frac{l^\epsilon}{\sqrt{l}} \left(\frac{l}{K^2} \right)^{s_1+1/4} K \quad \text{for } s_1 \geq 2.$$

It is sufficient to estimate

$$E := \frac{l^\epsilon}{\sqrt{l}} \sum_{n=1}^{l/2} \left(\frac{l}{n} \right)^{1/4} \left| \sum_k h\left(\frac{4k}{K}\right) \frac{\sin\left((4k-1) \arcsin \sqrt{n/l} - \pi/4\right)}{\sqrt{4k-1}} \right|.$$

Using the Poisson summation formula ([9, Theorem 4.4])

$$\sum_k h\left(\frac{4k}{K}\right) \frac{\sin\left((4k-1) \arcsin \sqrt{n/l} - \pi/4\right)}{\sqrt{4k-1}} = \sum_{m \in \mathbf{Z}} I(m),$$

where

$$I(m) := \int_{-\infty}^{+\infty} h\left(\frac{4y}{K}\right) \frac{\sin\left((4y-1)\arcsin\sqrt{n/l} - \pi/4\right)}{\sqrt{4y-1}} e^{-2\pi i y m} dy.$$

Let $g(y) := \frac{1}{4}Ky(-2\pi m \pm 4\arcsin\sqrt{n/l})$, then writing the sine in terms of exponentials we have

$$I(m) \ll K \left| \int_{-\infty}^{\infty} h(y) e^{ig(y)} \frac{dy}{\sqrt{yK-1}} \right|.$$

Integration by parts $a \geq 2$ times yields

$$I(m) \ll \sqrt{K} \left| \int_{-\infty}^{+\infty} \frac{\partial^a}{\partial y^a} \left(\frac{h(y)}{\sqrt{y-1/K}} \right) \frac{e^{ig(y)} dy}{K^a \left| -\pi m/2 \pm \arcsin\sqrt{n/l} \right|^a} \right|.$$

Note that $0 < \arcsin\sqrt{n/l} \leq \pi/4$. If $m \neq 0$ one has

$$I(m) \ll \sqrt{K}(Km)^{-a}$$

and

$$I(0) \ll \sqrt{K}(K \arcsin\sqrt{n/l})^{-a}.$$

Consequently,

$$\begin{aligned} E &\ll l^{-1/2+\epsilon} K^{1/2-a} \sum_{n=1}^{l/2} \left(\frac{l}{n}\right)^{1/4} \left(1 + \frac{1}{(\arcsin\sqrt{n/l})^a}\right) \ll \\ &l^{-1/2+\epsilon} K^{1/2-a} \sum_{n=1}^{l/2} \left(\frac{l}{n}\right)^{1/4+a/2} \ll l^{a/2-1/4+\epsilon} K^{-a+1/2}. \end{aligned}$$

□

Finally, we obtain the main result of this section.

Theorem 7.4. *For any $\epsilon > 0$, any $a \geq 2$, $l \ll K^2$ one has*

$$(7.8) \quad A_2(l) = \frac{2\tau(l)HK}{\sqrt{l}} \frac{1}{4} (2\log K - \log l - 2\log 8\pi + 2\gamma + 2H_1/H) + \\ O\left(K\left(\frac{l^{a/2-1/4+\epsilon}}{K^{a+1/2}} + \frac{l^\epsilon}{\sqrt{l}K} + \frac{l^{1/2+\epsilon}}{K^{7/2}}\right)\right).$$

The asymptotic formula for the averaged first moment follows from Theorem 3.1 .

Theorem 7.5. *There exist $c_1, c_2 > 0$ such that for $l \ll K$ one has*

$$(7.9) \quad A_1(l) = \frac{2}{\sqrt{l}} \frac{HK}{4} + O\left(\frac{K}{\sqrt{l}} \left(c_1 \frac{l}{K}\right)^{c_2 K}\right).$$

8. MOLLIFICATION AND NON-VANISHING AT THE CRITICAL POINT

8.1. The choice of mollifier. We choose a mollifier of type (see [10, 13])

$$(8.1) \quad M(f) = \sum_{m \leq M} x_m \lambda_f(m) m^{-1/2},$$

where

$$(8.2) \quad x_m = \frac{\mu(m)}{\rho(m)} P\left(\frac{\log M/m}{\log M}\right),$$

$$(8.3) \quad \rho(m) = \prod_{p|m} (1 + 1/p), \quad P(x) = x^2.$$

If M is not an integer ([13, Lemma 2.1])

$$(8.4) \quad \delta_{m < M} P\left(\frac{\log M/m}{\log M}\right) = \frac{2}{2\pi i} (\log M)^{-2} \int_{(3)} \frac{M^s}{m^s} \frac{ds}{s^3}.$$

Lemma 8.1. *Let $k \equiv 0 \pmod{2}$, $M = k^\Delta$. For any $\epsilon > 0$ there is $k_0 = k_0(\epsilon)$ such that for every $k \geq k_0$ the inequality*

$$(8.5) \quad \sum_{f \in H_{2k}(1)}^h M^2(f) \ll \log M$$

holds for any $\Delta < 1 - \epsilon$.

Proof. Consider

$$\sum_{f \in H_{2k}(1)}^h M^2(f) = \sum_{m_1, m_2 \leq M} \frac{x_{m_1} x_{m_2}}{\sqrt{m_1 m_2}} \sum_{f \in H_{2k}(1)}^h \lambda_f(m_1) \lambda_f(m_2).$$

The inner sum can be estimated using the Petersson trace formula and [23, Lemma 2.1]

$$\sum_{f \in H_{2k}(1)}^h \lambda_f(m_1) \lambda_f(m_2) = \delta(m_1, m_2) + O(e^{-k}) \text{ for } m_1 m_2 < \frac{k^2}{10^4}.$$

Note that $x_m \ll 1$, and, therefore,

$$\sum_{f \in H_{2k}(1)}^h M^2(f) \ll \sum_{m \leq M} \frac{1}{m} \ll \log M.$$

□

Averaging over k we prove the following estimate.

Lemma 8.2. *Let $M = K^\Delta$. For any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$ such that for every $K \geq K_0$ the inequality*

$$(8.6) \quad \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h M^2(f) \ll K \log M$$

holds for any $\Delta < 1 - \epsilon$.

8.2. The first mollified moment.

Lemma 8.3. *Let $k \equiv 0 \pmod{2}$ and $M = k^\Delta$. For any $\epsilon > 0$ there is $k_0 = k_0(\epsilon)$ such that for every $k \geq k_0$ one has*

$$(8.7) \quad M_1 := \sum_{f \in H_{2k}(1)}^h M(f) L_f(1/2) = \frac{4\zeta(2)}{\log M} + O((\log M)^{-2})$$

for any $\Delta < 1 - \epsilon$.

Proof. By Theorem 3.1

$$\sum_{f \in H_{2k}(1)}^h M(f) L_f(1/2) = 2 \sum_{m \leq M} \frac{x_m}{m} + O\left(\sum_{m \leq M} \frac{x_m}{m} \left(\frac{2\pi e m}{k}\right)^k\right).$$

Since

$$\sum_{m \leq M} \frac{x_m}{m} \left(\frac{2\pi e m}{k}\right)^k \ll (2\pi e M/k)^k = (2\pi e)^k k^{k(\Delta-1)},$$

the error term is negligible for any $\Delta < 1 - \epsilon$. Applying equations (8.2) and (8.4), we have

$$2 \sum_{m \leq M} \frac{x_m}{m} = \frac{4}{(\log M)^2} \frac{1}{2\pi i} \int_{(3)} \frac{M^s}{s^3} \sum_{m=1}^{\infty} \frac{\mu(m)}{\rho(m) m^{1+s}} ds.$$

Consider the sum over m

$$\sum_{m=1}^{\infty} \frac{\mu(m)}{\rho(m) m^{1+s}} = \frac{\alpha(s)}{\zeta(s+1)},$$

where

$$\alpha(s) = \prod_p \frac{1 + 1/p - 1/p^{s+1}}{(1 + 1/p)(1 - 1/p^{s+1})}$$

converges absolutely for $\Re s > -1$ and $\alpha(0) = \zeta(2)$. The resulting integral

$$\frac{4}{(\log M)^2} \frac{1}{2\pi i} \int_{(3)} \frac{M^s \alpha(s)}{s^3 \zeta(s+1)} ds$$

has a double pole at $s = 0$. We cross this pole by moving the contour of integration to

$$\Re s = -\frac{c}{\log(3 + |\Im s|)}.$$

Note that on the new contour $\zeta(s+1)$ has no zeros (see, for example, [26, Theorem 4, p. 33]) and the integral is bounded by

$$\int_0^\infty M^{-c/\log(3+t)} (3+t)^{-3+\epsilon} dt = \int_{\log 3}^\infty e^{x(-2+\epsilon)-cx^{-1}\log M} dx.$$

Using the saddle point method, we estimate the last integral as

$$(\log M)^{1/4} e^{-c'\sqrt{\log M}}.$$

Finally, the residue at $s = 0$ is equal to

$$\frac{4\zeta(2)}{\log M} + O((\log M)^{-2}).$$

□

Lemma 8.4. *Let $M = K^\Delta$. For any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$ such that for every $K \geq K_0$ one has*

$$(8.8) \quad A_1 := \sum_k h \left(\frac{4k}{K} \right) \sum_{f \in H_{4k}(1)}^h M(f) L_f(1/2) = \frac{HK}{4} \frac{4\zeta(2)}{\log M} + O(K(\log M)^{-2})$$

for any $\Delta < 1 - \epsilon$.

Proof. By Theorem 7.5 for some absolute constants $c_1, c_2 > 0$

$$A_1 = \frac{HK}{4} 2 \sum_{m \leq M} \frac{x_m}{m} + O \left(K \left(\frac{c_1 M}{K} \right)^{c_2 K} \right).$$

The error term is negligible for any $\Delta < 1 - \epsilon$. The main term was evaluated in Lemma 8.3. □

8.3. The second mollified moment.

Lemma 8.5. *Let $k \equiv 0 \pmod{2}$ and $M = k^\Delta$. For any $\epsilon > 0$ there is $k_0 = k_0(\epsilon)$ such that for every $k \geq k_0$ one has*

$$(8.9) \quad M_2 := \sum_{f \in H_{2k}(1)}^h M^2(f) L_f^2(1/2) = \frac{16\zeta^2(2)}{(\log M)^2} (1 + 1/\Delta) + O((\log M)^{-3})$$

for any $\Delta < 1/4 - \epsilon$.

Proof. Using the property of multiplicity (2.6) we have

$$\sum_{f \in H_{2k}(1)}^h M^2(f) L_f^2(1/2) = \sum_{b \leq M} \frac{1}{b} \sum_{m_1, m_2 \leq M/b} \frac{x_{m_1 b} x_{m_2 b}}{\sqrt{m_1 m_2}} M_2(m_1 m_2).$$

By Theorem 6.4 contribution of the error term in $M_2(m_1 m_2)$ is negligible for any $\Delta < 1/4 - \epsilon$. Indeed,

$$\sum_{b \leq M} \frac{1}{b} \sum_{m_1, m_2 \leq M/b} \frac{(m_1 m_2)^{1/2+\epsilon}}{\sqrt{m_1 m_2} k} \ll \frac{M^{2+\epsilon}}{\sqrt{k}} = k^{2\Delta-1/2+\epsilon}.$$

The main term of $M_2(m_1 m_2)$ is

$$2 \frac{\tau(m_1 m_2)}{\sqrt{m_1 m_2}} \left(2 \frac{\Gamma'}{\Gamma}(k) - \log(m_1 m_2) + 2\gamma - 2 \log(2\pi) \right).$$

Therefore, the largest contribution comes from

$$2 \frac{\tau(m_1 m_2)}{\sqrt{m_1 m_2}} \log \frac{k^2}{m_1 m_2}.$$

Therefore, we need to compute

$$2 \sum_{b \leq M} \frac{1}{b} \sum_{m_1, m_2 \leq M/b} \frac{\tau(m_1 m_2) x_{m_1 b} x_{m_2 b}}{m_1 m_2} \log \frac{k^2}{m_1 m_2}.$$

Using

$$\tau(m_1 m_2) = \sum_{d | (m_1, m_2)} \mu(d) \tau(m_1/d) \tau(m_2/d),$$

we have

$$2 \sum_{n \leq M} \frac{1}{n} \sum_{d | n} \frac{\mu(d)}{d} \sum_{m_1, m_2 \leq M/n} \frac{\tau(m_1) \tau(m_2) x_{m_1 n} x_{m_2 n}}{m_1 m_2} \log \frac{k^2}{d^2 m_1 m_2}.$$

The last expression splits into two parts:

$$P_1 := 4 \sum_{n \leq M} \frac{1}{n} \sum_{d|n} \frac{\mu(d)}{d} \sum_{m_1, m_2 \leq M/n} \frac{\tau(m_1)\tau(m_2)x_{m_1n}x_{m_2n}}{m_1m_2} \log k/m_1,$$

$$P_2 := -4 \sum_{n \leq M} \frac{1}{n} \sum_{d|n} \frac{\mu(d) \log d}{d} \sum_{m_1, m_2 \leq M/n} \frac{\tau(m_1)\tau(m_2)x_{m_1n}x_{m_2n}}{m_1m_2}.$$

The main contribution comes from P_1 due to the additional factor of $\log k/m_1$. By Cauchy's integral formula

$$\log k/m_1 = \frac{1}{2\pi i} \int_{C_\delta} \frac{k^z}{m_1^z z^2} dz,$$

where C_δ is a circle of radius δ around the point 0. Using (8.2) and (8.4), we have

$$P_1 = \frac{16}{(\log M)^4} \frac{1}{(2\pi i)^3} \int_{(3)} \int_{(3)} \int_{C_\delta} \sum_{n=1}^{\infty} \frac{\phi(n)}{n^{2+s_1+s_2}} M^{s_1+s_2} \times$$

$$\sum_{m_1, m_2=1}^{\infty} \frac{\tau(m_1)\tau(m_2)\mu(m_1n)\mu(m_2n)}{\rho(m_1n)\rho(m_2n)m_1^{s_1+z+1}m_2^{s_2+1}} \frac{k^z dz}{z^2} \frac{ds_1}{s_1^3} \frac{ds_2}{s_2^3}.$$

Let

$$\alpha_1(s) := \prod_p \frac{(1 + 1/p - 2p^{-s-1})(1 - 1/p)}{(1 - p^{-s-1})^2},$$

and

$$\beta_n(s) := \prod_{p|n} \frac{1 + 1/p}{1 + 1/p - 2p^{-s-1}}.$$

Then the sum over m_1 can be computed as follows

$$\sum_{m_1=1}^{\infty} \frac{\tau(m_1)\mu(m_1n)}{\rho(m_1n)m_1^{s_1+z+1}} = \frac{\mu(n)}{\rho(n)} \sum_{(m_1, n)=1} \frac{\tau(m_1)\mu(m_1)}{\rho(m_1)m_1^{s_1+z+1}} =$$

$$\frac{\mu(n)}{\rho(n)} \prod_{(p, n)=1} \left(1 - \frac{2}{p^{s_1+z+1}(1 + 1/p)} \right) =$$

$$\frac{\mu(n)}{\rho(n)} \frac{\zeta(2)}{\zeta^2(s_1 + z + 1)} \beta_n(s_1 + z) \alpha_1(s_1 + z).$$

Similarly,

$$\sum_{m_2=1}^{\infty} \frac{\tau(m_2)\mu(m_2n)}{\rho(m_2n)m_2^{s_2+1}} = \frac{\mu(n)}{\rho(n)} \frac{\zeta(2)}{\zeta^2(s_2 + 1)} \beta_n(s_2) \alpha_1(s_2).$$

Now the sum over n is equal to

$$\begin{aligned} & \sum_{n=1}^{\infty} \frac{\phi(n)}{n^{2+s_1+s_2}} \frac{\mu^2(n)}{\rho^2(n)} \beta_n(s_1+z) \beta_n(s_2) = \\ & \sum_{n=1}^{\infty} \frac{\phi(n) \mu^2(n)}{n^{2+s_1+s_2}} \prod_{p|n} (1 + 1/p - 2p^{-s_1-z-1})^{-1} (1 + 1/p - 2p^{-s_2-1})^{-1} = \\ & \prod_p \left(1 + \frac{(1 - \frac{1}{p})(1 + \frac{1}{p} - 2p^{-s_1-z-1})^{-1}}{p^{s_1+s_2+1}(1 + \frac{1}{p} - 2p^{-s_2-1})} \right) = \zeta(s_1+s_2+1) \alpha_2(s_1, s_2, z), \end{aligned}$$

where

$$\alpha_2(s_1, s_2, z) = \prod_p \left(1 + \frac{(1 - \frac{1}{p})(1 + \frac{1}{p} - \frac{2}{p^{s_1+z+1}})^{-1}}{p^{s_1+s_2+1}(1 + \frac{1}{p} - \frac{2}{p^{s_2+1}})} \right) \left(1 - \frac{1}{p^{s_1+s_2+1}} \right).$$

Let us denote

$$\alpha(s_1, s_2, z) := \alpha_1(s_1+z) \alpha_1(s_2) \alpha_2(s_1, s_2, z).$$

Note that $\alpha(s_1, s_2, z)$ converges absolutely for $\Re s_1, \Re s_2, \Re z > -\epsilon$ for some $\epsilon > 0$ and $\alpha(0, 0, 0) = 1$. As a result,

$$\begin{aligned} P_1 &= \frac{16\zeta^2(2)}{(\log M)^4} \frac{1}{(2\pi i)^3} \int_{(3)} \int_{(3)} \int_{C_\delta} M^{s_1+s_2} \alpha(s_1, s_2, z) \times \\ & \quad \frac{\zeta(s_1+s_2+1)}{\zeta^2(s_1+z+1)\zeta^2(s_2+1)} \frac{k^z dz}{z^2} \frac{ds_1}{s_1^3} \frac{ds_2}{s_2^3}. \end{aligned}$$

Let γ_i denote the line

$$\Re s_i = -\frac{c}{\log(3 + |\Im s_i|)},$$

$$f(s_1, s_2, z) := M^{s_1+s_2} \alpha(s_1, s_2, z) \frac{\zeta(s_1+s_2+1)}{\zeta^2(s_1+z+1)\zeta^2(s_2+1)} \frac{k^z}{z^2 s_1^3 s_2^3}.$$

First, we evaluate the integral over z , getting

$$\begin{aligned} I &:= \frac{1}{(2\pi i)^3} \int_{(3)} \int_{(3)} \int_{C_\delta} f(s_1, s_2, z) dz ds_1 ds_2 = \\ & \quad \frac{1}{(2\pi i)^2} \int_{(3)} \int_{(3)} \operatorname{res}_{z=0} f(s_1, s_2, z) ds_1 ds_2. \end{aligned}$$

Moving the contour of integration to the line γ_1 , we cross a pole at $s_1 = 0$. By Sokhotski-Plemelj Theorem

$$I = \frac{1}{2\pi i} \int_{(3)} \operatorname{res}_{\substack{s_1=0 \\ z=0}} f(s_1, s_2, z) ds_2 + \frac{1}{2\pi i} \int_{\gamma_1} \operatorname{res}_{\substack{s_2=-s_1 \\ z=0}} f(s_1, s_2, z) ds_1 \\ + \frac{1}{(2\pi i)^2} \int_{\gamma_1} \int_{(3)} \operatorname{res}_{z=0} f(s_1, s_2, z) ds_2 ds_1.$$

Next we move the contour of integration over s_2 to the line γ_2 . Thus

$$I = \frac{1}{2\pi i} \int_{\gamma_1} \operatorname{res}_{\substack{s_2=0 \\ z=0}} f(s_1, s_2, z) ds_1 + \frac{1}{2\pi i} \int_{\gamma_2} \operatorname{res}_{\substack{s_1=0 \\ z=0}} f(s_1, s_2, z) ds_2 + \\ \frac{1}{(2\pi i)^2} \int_{\gamma_1} \int_{\gamma_2} \operatorname{res}_{z=0} f(s_1, s_2, z) ds_2 ds_1 + \\ \frac{1}{2\pi i} \int_{\gamma_1} \operatorname{res}_{\substack{s_2=-s_1 \\ z=0}} f(s_1, s_2, z) ds_1 + \operatorname{res}_{s_1=s_2=z=0} f(s_1, s_2, z).$$

The contribution of the first three integrals above is negligible and can be estimated similarly to the proof of Lemma 8.3. The fourth integral can be bounded by a constant and, therefore, its contribution to P_1 is $O((\log M)^{-4})$. The main term is given by the residue at $s_1 = s_2 = z = 0$. The function $f(s_1, s_2, z)$ has a simple pole at $s_2 = 0$. Hence

$$\operatorname{res}_{s_1=s_2=z=0} f(s_1, s_2, z) = 2\pi i \operatorname{res}_{s_1=z=0} \frac{k^z M^{s_1} \zeta(s_1 + 1) \alpha(s_1, 0, z)}{z^2 s_1^3 \zeta^2(s_1 + z + 1)}.$$

Next, we compute the residue at $z = 0$, where the resulting function has a double pole. Finally, evaluating the residue at the triple pole $s_1 = 0$ we find that

$$P_1 = \frac{16\zeta^2(2)}{(\log M)^4} (\log k \log M + (\log M)^2) + O((\log M)^{-3}) = \\ \frac{16\zeta^2(2)}{(\log M)^2} (\Delta^{-1} + 1) + O((\log M)^{-3}).$$

Similarly, using the representation

$$\log d = \frac{1}{2\pi i} \int_{C_\delta} \frac{dz}{z^2},$$

we prove that $P_2 = O((\log M)^{-3})$. □

Lemma 8.6. *Let $M = K^\Delta$. For any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$ such that for every $K \geq K_0$ one has*

$$(8.10) \quad A_2 := \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h M^2(f) L_f^2(1/2) = \frac{HK}{4} \frac{16\zeta^2(2)}{(\log M)^2} (1 + 1/\Delta) + O(K(\log M)^{-3})$$

for any $\Delta < 1 - \epsilon$.

Proof. Consider

$$A_2 = \sum_{b \leq M} \frac{1}{b} \sum_{m_1, m_2 \leq M/b} \frac{x_{m_1 b} x_{m_2 b}}{\sqrt{m_1 m_2}} A_2(m_1 m_2),$$

where the asymptotics of $A_2(m_1 m_2)$ is given by Theorem 7.4. Accordingly, contribution of the error term is bounded by

$$KM^\epsilon \left(\frac{M^{a+1/2}}{K^{a+1/2}} + \frac{1}{\sqrt{K}} + \frac{M^2}{K^{7/2}} \right)$$

for any $a \geq 2$. This is negligible if $\Delta < 1 - \epsilon$. The main term can be evaluated similarly to Lemma 8.5. \square

8.4. Non-vanishing for an individual weight.

Theorem 8.7. *For any $\epsilon > 0$ there exists $k_0 = k_0(\epsilon)$ such that for any $k \geq k_0$ and $k \equiv 0 \pmod{2}$ we have*

$$(8.11) \quad \sum_{\substack{f \in H_{2k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^h 1 \geq \frac{1}{5} - \epsilon.$$

Proof. Asymptotics of the first and second mollified moments is given by equations (8.7) and (8.10). Accordingly, the largest admissible

length of mollifier is $\Delta < 1/4 - \epsilon$. Applying inequality (8.5), we estimate

$$\begin{aligned} \widetilde{M}_1 &:= \sum_{f \in H_{2k}(1)}^h M(f) L_f(1/2) \delta_{L_f(1/2) < b(k)(\log k)^{-1/2}} \leq \\ &\left(\sum_{\substack{f \in H_{2k}(1) \\ L_f(1/2) < b(k)(\log k)^{-1/2}}}^h L_f^2(1/2) \right)^{1/2} \left(\sum_{f \in H_{2k}(1)}^h M^2(f) \right)^{1/2} \leq \\ &b(k)(\log k)^{-1/2} \left(\sum_{f \in H_{2k}(1)}^h M^2(f) \right)^{1/2} \leq b(k). \end{aligned}$$

Taking $b(k) = (\log k)^{-3/2}$ we have

$$\sum_{\substack{f \in H_{2k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^h 1 \geq \frac{(M_1 - \widetilde{M}_1)^2}{M_2} \geq \frac{\Delta}{1 + \Delta}$$

for any $\Delta < 1/4 - \epsilon$. The result follows. \square

8.5. Non-vanishing on average.

Theorem 8.8. *For any $\epsilon > 0$ there is $K_0 = K_0(\epsilon)$ such that for any $K \geq K_0$ we have*

$$(8.12) \quad \frac{4}{HK} \sum_k h\left(\frac{4k}{K}\right) \sum_{\substack{f \in H_{4k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^h 1 \geq \frac{1}{2} - \epsilon.$$

Proof. Note that

$$\sum_k h\left(\frac{4k}{K}\right) \sim \frac{HK}{4} \text{ as } K \rightarrow \infty.$$

The Cauchy-Schwartz inequality and estimate (8.6) yield

$$\widetilde{A}_1 := \sum_k h\left(\frac{4k}{K}\right) \sum_{\substack{f \in H_{4k}(1) \\ L_f(1/2) < b(k)(\log k)^{-1/2}}}^h M(f) L_f(1/2) \ll Kb(k).$$

Choosing $b(k) = (\log k)^{-3/2}$ and applying the Cauchy-Schwartz inequality twice, we obtain

$$\begin{aligned} & \sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h L_f(1/2) \delta_{L_f(1/2) \geq (\log k)^{-2}} \leq \\ & \sum_k h\left(\frac{4k}{K}\right) \sqrt{\sum_{f \in H_{4k}(1)}^h L_f^2(1/2)} \sqrt{\sum_{f \in H_{4k}(1)}^h \delta_{L_f(1/2) \geq (\log k)^{-2}}} \leq \\ & \left(\sum_k h\left(\frac{4k}{K}\right) \sum_{f \in H_{4k}(1)}^h L_f^2(1/2) \right)^{1/2} \left(\sum_k h\left(\frac{4k}{K}\right) \sum_{\substack{f \in H_{4k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^h 1 \right)^{1/2}. \end{aligned}$$

Therefore, by Lemmas 8.4 and 8.6 one has

$$\frac{4}{HK} \sum_k h\left(\frac{4k}{K}\right) \sum_{\substack{f \in H_{4k}(1) \\ L_f(1/2) \geq (\log k)^{-2}}}^h 1 \geq \frac{\left(\frac{4}{HK} A_1 - \frac{4}{HK} \tilde{A}_1\right)^2}{\frac{4}{HK} A_2} \geq \frac{\Delta}{1 + \Delta}$$

for any $\Delta < 1 - \epsilon$. □

ACKNOWLEDGEMENTS

The authors would like to thank Viktor A. Bykovskii and the Institute for Applied Mathematics in Khabarovsk for hospitality and excellent working conditions. We are grateful to Guillaume Ricotta for careful reading of the manuscript and helpful comments. We also thank Philippe Michel and Emmanuel Royer for encouraging discussions.

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